Math 129-11  Exam 1 Solutions, 5-6 (from one version of exam)

None of these problems requires a calculator for doing the integration. In all cases, in order to receive full credit, you should show enough work that it is clear that you can do the problem correctly and completely without using a calculator [emphasis added]. Of course, there is nothing wrong with using calculator as permitted by course policy to CHECK an answer.

5. Without using integral formulas or techniques, verify (check) the following two formulas and state whether the formula given is correct or not.

(a) \[ \int \sec(x) \, dx = \ln \left[ \sec(x) + \tan(x) \right] + C \]

(b) \[ \int \arccos(x) \, dx = (1 - x^2)^{1/2} + C \]

COMMENT. The basic idea of antiderivatives is that
\[ \int f(x) \, dx = F(x) + C \text{ if and only if } F'(x) = f(x). \]

This, along with the Fundamental Theorem of Calculus, is the FUNDAMENTAL idea of calculus.

There are TECHNIQUES (discussed in Chapter 7, for example) for finding antiderivatives, but the fundamental idea is that antiderivatives are the “opposite” of derivatives.

So, to CHECK (verify) an antiderivative, one DIFFERENTIATES the answer; one does not, at least at first, attempt to find some technique (e.g., from Chapter 7) which gets you to the answer.

This has been discussed often in class, since the first day of class.

SOLUTION. For typing convenience, we will use \( D_x \) to stand for \( \frac{d}{dx} \).

(a) Using the rule for differentiating the natural logarithm, the chain rule, and the rules for differentiating trig functions (from first semester calculus),
\[
D_x \ln \left[ \sec(x) + \tan(x) \right] = \left[ \frac{1}{\left[ \sec(x) + \tan(x) \right]} \right] D_x \left[ \sec(x) + \tan(x) \right] \\
= \left[ \frac{1}{\left[ \sec(x) + \tan(x) \right]} \right] \cdot \sec(x) \tan(x) + \sec^2(x) \\
= \left[ \frac{\sec(x) \tan(x) + \sec^2(x)}{\sec(x) + \tan(x)} \right] \cdot \sec(x) \\
= \sec(x) \text{ (the first factor and second factor “cancel”).}
\]

Since the derivative of \( \ln \left[ \sec(x) + \tan(x) \right] \) is \( \sec(x) \),
\[ \int \sec(x) \, dx = \ln \left[ \sec(x) + \tan(x) \right] + C, \] as we wished to verify.

COMMENT. It is not necessary for your solution to contain this many details. It is obvious, however, that some students don’t know how to use the chain rule or to differentiate these trig functions, so the details are given here.

(b) Using the chain rule to differentiate the expression with the \( 1/2 \) exponent, we get
\[ D_x (1 - x^2)^{1/2} = \frac{1}{2}(1 - x^2)^{-1/2} \cdot (-2x) = -x (1 - x^2)^{-1/2} \]

This is NOT \( \arccos(x) \) -- NO WAY.

[E.g., when \( x = 1 \), \( \arccos(x) = \arccos(1) = 0 \), whereas \( -x (1 - x^2)^{-1/2} \) is undefined when \( x = 1 \).

Similarly, when \( x = 0 \), \( \arccos(x) = \arccos(0) = \pi/2 \), whereas \( -x (1 - x^2)^{-1/2} \) is zero when \( x = 0 \).]

6. Evaluate \( \int x \sec(x^2) \, dx \) without using a calculator or other “outside” sources.

SOLUTION (more details elsewhere). Substitute \( u = x^2 \), and then use 5(a) above!