PART I.

Suppose we have a power series, which for simplicity in thinking and writing we will assume is about \( x = 0 \). (Later, a discussion of results for a power series about \( x = a \).)

\[
C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots
\]

Here’s the first important point: Since we think of the coefficients \( C_n \) as “constants” and the point \( a \) as a “constant” (\( a = 0 \) in this simple case), and we think of \( x \) as a variable, we can think of this power series as a function.

\[
f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \ldots
\]

Here’s the second important point, just a collection of calculations. Let’s use elementary calculus to calculate the derivatives of \( f(x) \) at \( x = 0 \). (The details of this were done in class, and you should be able to do it yourself.)

For the function itself, from \( f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \ldots \) we get

\[
f(0) = C_0.
\]

From \( f'(x) = C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + \ldots \) we get

\[
f'(0) = C_1, \text{ which can be written } f'(0) = 1! C_1. \text{ (The reason for doing this will be seen soon.)}
\]

From \( f''(x) = 2C_2 + 3\cdot2C_3 x + 4\cdot3C_4 x^2 + \ldots \) we get

\[
f''(0) = 2C_2, \text{ which can be written } f''(0) = 2! C_2. \text{ (If we’re thinking, we begin to see a pattern.)}
\]

From \( f'''(x) = 3\cdot2C_3 + 4\cdot3\cdot2C_4 x + \ldots \) we get

\[
f'''(0) = 3\cdot2C_3, \text{ which can be written } f'''(0) = 3! C_3. \text{ (If we’re thinking, we see the pattern.)}
\]

Here’s the final point for this First Part: Continuing in this way, we can guess (and it can be confirmed) that the \( n \)-th derivative at 0 is

\[
f^{(n)}(0) = n! C_n.
\]

This is a fundamental relation between the coefficients in the power series, \( C_n \), and the function, \( f(x) \), which is the sum of the power series.

To be continued with PART II.