Math 323 (Laetsch)  Inverse Functions  21st Century

Another version of the discussion below, with some more detail, may be posted online under Announcements on the class website.

**In class, we defined the inverse of a function** by thinking of the function as a relation and defining the inverse function to be the inverse relation.

In class, before attacking the idea of the inverse of a function, we began with a general relation; we defined in general the inverse relation. Given a relation \( R \), the inverse relation (for this given relation \( R \)) is the relation obtained by reversing all the ordered pairs in the original relation \( R \). (In class, we denoted the inverse by putting a “bar” over the relation. (So, obviously, for every relation there is a corresponding inverse relation; there is an important case when the inverse relation is the same as the original, and you should be able to determine this.)

Then we stated

if the given relation is a function from a set \( A \) to a set \( B \), then the following are equivalent:

(a) The original function is a bijection from \( A \) to \( B \) (i.e., it is bijective).

(b) The inverse relation is a function from \( B \) to \( A \).

[We are not saying that (a) and (b) are each TRUE. We are saying that they are EQUIVALENT: IF one is true, then so is the other.]

In other words,

If \( f : A \to B \), then

the inverse relation is a function from \( B \) to \( A \) iff \( f : A \to B \) is bijective.

When talking about functions which have an inverse function, by this definition there is obviously only one such inverse function, and it is denoted, as usual, by \( f^{-1} : B \to A \).

It is also true that when we start with a bijection from \( A \) to \( B \), then the inverse function is a bijection from \( B \) to \( A \).

[You should understand all of what was just said, and you should try to prove as much of it as possible; all the proofs should be within the capability of Math 323 students.]

An alternative way of looking at inverse functions, similar to what is done in the textbook, starts with a function in general and uses an existential approach instead of a constructive approach as we did above.

Given \( f : A \to B \), we suppose there exists

\( g : B \to A \) such that \( g \circ f \) is the identity function on \( A \) and \( f \circ g \) is the identity function on \( B \), i.e.,

For all \( x \) in \( A \), \( g(f(x)) = x \), and for all \( x \) in \( B \), \( f(g(x)) = x \).

We may temporarily refer to this as an “algebraic inverse”, just to distinguish it temporarily from the inverse as defined above using relations.

Facts: Such an “algebraic inverse” function, if it exists, is unique (there is only one such \( g \)) and if such a \( g \) exists, then the original \( f : A \to B \) is a bijection. Further, when \( f : A \to B \) is a bijection, then the algebraic inverse as just defined is in fact the same as the inverse function \( f^{-1} : B \to A \) defined above using relations (the uniqueness is a consequence of this). **You should be able to prove all of this**, even if you haven’t been shown all the steps in the proofs. These are all ideas which Math 323 students should be able to handle and to prove.