Math 323  Algebra and Functions  21st century

Algebra.

1. **Prove** (correctly, completely, clearly, carefully, and concisely) that for every real number \( y \) in the interval \([3, 19]\), there is a real number \( x \) in the interval \([-4, 2]\) such that \( y = x^2 + 3 \).
   (Of course, you should use only algebra in the proof. And, of course, relying on a graph for your proof, or making unproved assertions about maxima and minima, is unacceptable; a graph can be useful to help you decide how to do the proof, but it is not part of the proof.
   **Keep these remarks in mind throughout the rest of the course.**)

Functions. (As always, check the specific Lesson online for detailed instructions.)

2. Determine the range of the function \( f: \mathbb{R} \to \mathbb{R} \) given by \( f(x) = (x + 3)^2 - 4 \). Illustrate with a graph.
   Proof MIGHT not necessary for this assignment, but you should be able to prove your answer.

3. Determine the range of the function \( f: \mathbb{R} \to \mathbb{R} \) given by \( f(x) = x^2 + 6x + 5 \). Illustrate with a graph.
   Proof MIGHT not necessary for this assignment, but you should be able to prove your answer.

4. Determine the range of the function \( f: [-4, 2] \to \mathbb{R} \) given by \( f(x) = x^2 + 3 \). Illustrate with a graph.
   Proof MIGHT not necessary for this assignment, but you should be able to prove your answer.

5. Suppose \( f: X \to Y \), with domain \( X \). Prove:
   If there exists a function \( g: Y \to X \), with domain \( Y \), such that for all \( x \) in \( X \), \( g(f(x)) = x \),
   then \( f \) is injective*. (The function \( g \) is called a “left inverse” of \( f \). More precisely, note that the domain of \( g \) is given to be the codomain of \( f \), and the codomain of \( g \) is chosen to be the domain of \( f \).)

6. **REFER to the function \( f \) defined in Problem 11.7 in Section 11.5 in the textbook, given by \( f(x) = x^2 - 6x + 3 \) for all \( x \) in the interval \([3, \infty)\).**

   (prelude) The book claims that this defines a function from \([3, \infty)\) to \([-6, \infty)\). **Verify** this claim.
   [Since \( f(x) \) is clearly defined for all \( x \) in \([3, \infty)\), this is clearly an acceptable domain. Worry about the codomain.]
   (a) **Do part (a)** in Problem 11.7 and prove your answer.
   (b) **Do part (b)** in Problem 11.7.*

*Comment on “injective”.

For a function, “injective” means the same as “one-to-one” – different inputs have different outputs.
I.e., for a function \( f: \) For all \( x \) and \( y \) in the domain of \( f \), if \( x \neq y \), then \( f(x) \neq f(y) \).
To prove that a function is injective, it is usually much more convenient to prove the contrapositive,

\[
\text{for all } x \text{ and } y \text{ in the domain of } f, \text{ if } f(x) = f(y), \text{ then } x = y.
\]

This is the “standard” way of proving that a function is injective, but of course that does not mean you “have to” do it this way if there is an easier way to do it; in particular, if you can UPR, do so. (It’s the idea of “correct, complete, clear, careful, and concise” again.)