7. Induction

Use the Principle of Mathematical Induction where relevant to prove that

there exists a natural number \( m \) such that for all natural numbers \( n \geq m \), \( 3(2^n) \leq n! \)

Give a careful proof, following the guidelines given in this course for doing induction proofs.

Proof.  

As usual with induction-related proofs, we begin the by defining the “statement function” for the fact we want to prove.

For each natural number \( n \), let \( P(n) \) be the statement \( 3(2^n) \leq n! \).

We are asked to prove “there exists \( m \) ...”.

The usual way to do such a proof is to do some thinking to find \( m \), exhibit \( m \), and prove that it has the properties claimed.

Choose \( m = 10 \). We claim that \( P(n) \) is true for all \( n \geq m \).

Proof by induction:

Base case, \( n = 10 \). We want to prove \( P(10) : 3(2^{10}) \leq 10! \)

Since \( 2^{10} = 1024 \) and \( 10! = 3628800 \),

\( 3(2^{10}) = 3(1024) = 3072 \leq 3628800 = 10! \). Thus \( 3(2^{10}) \leq 10! \),

so \( P(10) \) is true.

Inductive step:

Assume \( P(n) \) is true for some \( n \geq 10 \), so \( 3(2^n) \leq n! \)

[Comment: The step above is essential; it is the heart of the induction proof.

One assumes \( P(n) \) is true and proves \( P(n+1) \) is true for appropriate values of \( n \).]

We want to prove \( P(n+1) : 3(2^{n+1}) \leq (n+1)! \)

We start with the Left Hand Side and work our way to the Right Hand Side.

We do NOT start by assuming what we want to prove!

\[
3(2^{n+1}) = 3(2^n)(2) \leq n!(2) \quad \text{by the Inductive Hypothesis} \quad P(n)
\]

\[
= 2(n!)
\]

\[
\leq (n+1)(n!) \quad \text{since} \quad n \geq 10, \quad \text{so} \quad n+1 \geq 2
\]

\[
= (n+1)!
\]

Thus \( 3(2^{n+1}) \leq (n+1)! \), so \( P(n+1) \) is true.

Summary: We have proved the base case, \( P(10) \), and we have proved, for all \( n \geq 10 \),

if \( P(n) \) is true, then \( P(n+1) \) is true.

By PI, it follows that \( P(n) \) is true for all \( n \geq 10 \).