8. This problem is about the Division Algorithm. This theorem has an “existence” part and a “uniqueness” part. For the purposes of this problem, you can focus on existence and ignore uniqueness. The first problem below asks you to state the Division Algorithm for a certain case. Then the second problem asks you to USE this result just stated to prove the Division Algorithm for another case.

(a) Suppose \( n \) and \( d \) natural numbers. What does the Division Algorithm tell you about the existence of a ”quotient” and a “remainder”? [“There exist ... ”] State the relevant equalities and inequalities.

(b) Suppose \( n \) is a negative integer and \( d \) is a natural number. Suppose \( d \) does NOT divide \( n \). State and prove the Division Algorithm (the existence part of the Division Algorithm) for this case, using the fact just stated in (a) (the Division Algorithm for natural numbers).

**SOLUTIONS** (ignoring uniqueness):

(a) There exist integers \( q \) (quotient) and \( r \) (remainder) such that \( n = qd + r \) and \( 0 \leq r < d \).

(b) There exist integers \( q \) and \( r \) such that \( n = qd + r \) and \( 0 < r < d \). \((r > 0 \text{ because } d \text{ does not divide } n.)\)

Proof. Assume \( n \) is a negative integer and \( d \) is a natural number. Then \( -n \) is a natural number, so by (a), there exist integers \( q' \) and \( r' \) such that \(-n = q'd + r', \) where \( 0 < r' < d \) \((r' > 0 \text{ since } d \text{ does not divide } n \text{ and therefore does not divide } -n).\)

So, \( n = -q'd - r' = (-q' - 1)d + d - r' = qd + r, \) for \( q = -q' - 1 \) and \( 0 < r = d - r' < d \).

This proves the statement in (b) \([\text{with } q = -q' - 1 \text{ and } r = d - r'].\)