Final Exam, Solution, Problems 9 and 11

Remember that notation \( f : A \rightarrow B \) ALWAYS implies that \( A = \text{dom}(f) \)
(in this class, and in many, but not all, classes).

8. [Included for reference] Give a complete proof that the open intervals \((0, 1)\) and \((1, \infty)\) are equinumerous.
   (Optional hint: \(1/x\)) [Proof not given here.]

9. We showed in class how to prove that any two bounded open intervals are equinumerous; assume this fact. Prove, using the basic properties of the “equinumerous” relation, that if \( J \) is a bounded open interval, then \( J \) and \((1, \infty)\) are equinumerous.

   **Proof.** [HDYSP?] Consider a bounded open interval \( J \). By our assumption on bounded open intervals, \( J \) and \((0, 1)\) are equinumerous, since \((0, 1)\) is a bounded open interval. By Problem 8, \((0, 1)\) and \((1, \infty)\) are equinumerous. Since “equinumerous” is a transitive relation, \( J \) and \((0, \infty)\) are equinumerous.

   **Very Important Comment.**

   \((1, \infty)\) is not a BOUNDED interval.

   It is bounded BELOW, but in order for a set of real numbers to be “bounded”, it must be bounded ABOVE AND bounded BELOW.

11. Throughout the rest of this problem, and in your solutions, assume that all lower case letters such as \( x, y, \) etc., refer to real numbers.

   Decide whether the statement given below (\( \rightarrow \)) is TRUE or FALSE.

   If true prove.

   If false, prove or explain clearly.

   You may use standard algebraic facts about the exponential function, e.g., for all \( x, \ e^x > 0. \)

   \( \rightarrow \) Assume \( f : \mathbb{R} \rightarrow \mathbb{R} \). Then for all \( x, \) there exist \( c \) such that \( f(x) = c \ e^x. \)

   **Comment.** A very simple and basic “for all ... there exists ... ” problem.

   **SOLUTION.** TRUE.

   Proof. [HDYSP?] Assume \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( x \in \mathbb{R} \).

   [How do you prove “there exists”? Search for \( c \) (in this case) which does the job.] Since \( e^x \) is never 0, we can choose \( c = f(x) \ e^{-x}. \)

   Then \( f(x) = c \ e^x \), as desired.

   **COMMENTS.**

   1. There is no need to replace \( f(x) \) by \( y; \) doing so is a waste of time and suggests a misunderstanding of function notation.

   2. The point is not to do the proof for YOUR choice of function \( f, \) e.g., \( f(x) = 0 \) or \( f(x) = e^x. \) The point is to prove this for an “arbitrary” function \( f. \)

   3. VERY IMPORTANT. The last equation in your proof should not be an inconsequential triviality like \( f(x) = f(x) \) or a redundancy like \( f(x) = c \ e^x = f(x). \) You have found you \( c; \) then the point is \( f(x) = c \ e^x. \)