Functions.

2. Assume \( g : (-\infty, 2) \to \mathbb{R} \) is given by

\[
\text{For each } t \text{ in } (-\infty, 2), \quad g(t) = |t| + 2.
\]

Find the range of \( g \) and prove your answer, following the guidelines given in this class for proving such statements (e.g., use only simple algebra).

**SOLUTION.** The range of \( g \) is \([2, \infty)\).

Proof. Suppose \( y \) is in \( \text{ran}(g) \). Then there exists \( t \) in \( \text{dom}(g) \) such that

\[
y = g(t) = |t| + 2 \geq 2,
\]

so \( y \) is in \([2, \infty)\).

Conversely, suppose \( y \) is in \([2, \infty)\), so \( y \geq 2 \) (NOT \( 2 \leq y < \infty \)! See Project I.)

Choose \( t = 2 - y \); this is negative since \( y \geq 2 \). So, \( t \) is in \( \text{dom}(g) = (-\infty, 2) \), and

\[
g(t) = |t| + 2 = -(2 - y) + 2 = y,
\]

so \( y \) is in \( \text{ran}(g) \).

This proves that \( \text{ran}(g) = [2, \infty) \).

3. Find a new codomain \( B \) for the function \( g \) given in the preceding problem (Problem 2) so that \( g : (-\infty, 2) \to B \) is surjective and explain your answer.

**SOLUTION.** Choose the codomain \( B \) to be the range, \([2, \infty)\). Since the codomain is the range, the function \( g : (-\infty, 2) \to B \) is surjective.

Order.

5. Let \( S \) be a set of real numbers; suppose \( S \) has a maximum \( m \).

(You should know what a “maximum” is.).

Prove that \( m \) is the supremum of the set \( S \) in TWO different ways:

a. Using the “least upper bound” definition of supremum.

b. Using the “approximation property” of supremum.

You don’t need the Completeness Axiom.

**SOLUTION.**

a. Assume \( S \) has a maximum \( m \). To prove that \( m = \sup(S) \): By definition of maximum, \( m \) is an upper bound of \( S \) and \( m \) is an element of \( S \). So, as just stated, \( m \) is an upper bound of \( S \).

Suppose \( b \) is an upper bound of \( S \). Since \( m \) is an element of \( S \), \( b \geq m \). So, \( m \) is the least upper bound of \( S \).

b. As above, \( m \) is an upper bound of \( S \) and \( m \) is an element of \( S \). To use the approximation property, suppose \( x < m \). Then, trivially, \( m \) is an element of \( S \) and \( m > x \). So, by the approximation property for suprema, \( m - \sup(S) \).