Archimedean property.

**The set of natural numbers is unbounded above.**

Another example of a proof by contradiction.

First, recall the approximation property:

Suppose \( m \) is an upper bound of a set \( S \).
Then \( m = \sup(S) \) iff

- for all \( x < m \), there exists \( s \) in \( S \) such that \( s > x \).

[BWOC] Suppose \( \mathbb{N} \) is bounded above.

Then \( \mathbb{N} \) has a supremum, say \( m \).
We will get a contradiction.
Think about \( m - 1 \).
As we all know, \( m - 1 < m \).
Apply approximation property:

- There exists \( n \) in \( \mathbb{N} \) such that \( n > m - 1 \).

So \( n + 1 > m = \sup(\mathbb{N}) \). Of course, since \( n \in \mathbb{N} \), then \( n + 1 \in \mathbb{N} \),

Contradiction!

Thus, \( \mathbb{N} \) is not bounded above.

\( \mathbb{N} \) is bounded above means

- There exists \( b \) in \( \mathbb{R} \) such that for all \( n \) in \( \mathbb{N} \), \( n \leq b \).

\[ \exists b \text{ in } \mathbb{R} \text{ s.t. } \forall n \text{ in } \mathbb{N}, \; n \leq b. \]

This is not true, so

\[ \forall x \text{ in } \mathbb{R}, \; \exists n \text{ in } \mathbb{N} \text{ s.t. } n > x. \]

For all \( x \) in \( \mathbb{R} \), there exists \( n \) in \( \mathbb{N} \) such that \( n > x \).
Archimedian Property:

For every real number \( x \), there exists a natural number \( n \) such that \( n > x \).

One consequence.

Suppose \( x \) is a real number. Then there exists a smallest integer which is \( \geq x \). This number is sometimes called the ceiling of \( x \) and denoted by \( \lceil x \rceil \).

For simplicity, we real prove this only for positive numbers \( x \) (so that the ceiling is a natural number).

Proof. First, we show there is a natural number \( \geq x \). Then, we show there is a smallest one (least, minimum).

Why is there a natural number \( \geq x \)?

So, consider the set \( \{ n \in \mathbb{N} : n \geq x \} \).

Why does this set have a least element?

So, for every positive number \( x \), there is a smallest natural number \( n \geq x \).

Examples: \( \lceil \pi \rceil = ? \quad \lceil 7 \rceil = ? \)
Variation:  
For every number $x \geq 0$, there is a smallest natural number $n > x$.  
Use the same proof, or UPR.

Simple generalizations and variations: Can replace “positive” by “real” and “natural number” by “integer”.

Can also do it on the left: 
Suppose $x$ is a real number. Then there exists a largest integer which is $\leq x$. This number is sometimes called the floor of $x$ and denoted by $\lfloor x \rfloor$. For example, $\lfloor \pi \rfloor = ?$, $\lfloor 7 \rfloor = ?$

We will specifically make use of the following: 
For every $x \geq 0$, there exists a smallest natural number $n > x$.

We use this to prove 
The density of the rationals: 
Between any two distinct real numbers, there is a rational number. 
More precisely, for any two distinct real numbers, there is a rational number strictly between them.

If you prefer algebra to words: 
If $a$ and $b$ are real numbers with $a < b$, then there is a rational number $r$ with $a < r < b$.

As usual with universally quantified statements like this, it is one of the extreme sides of the statement which is important. In this case, if $a$ and $b$ are far apart, finding a rational number between them is not a big deal (in fact, we can find an integer between them).

Between any two real numbers, NO MATTER HOW CLOSE THEY ARE, there is a rational number.