NOTES: Math 323-2 April 23, 2021  Completeness Axiom,

As pointed out earlier, you should be able to give precise definitions of
• lower bound
• bounded below
• minimum
• infimum (greatest lower bound)
• approximation property for infimum

(If \( m \) is a lower bound of \( S \), then \( m \) is \( \text{inf}(S) \) iff for all \( x > m \), there exists \( s \) in \( S \) with \( s < x \).)

So here’s the “left-handed Completeness Axiom” (this is not standard terminology):
Every set of real numbers which is nonempty and bounded below has an infimum.

This is not something new. It can be proved using the Completeness Axiom as discussed above (on the right hand). There are two ways to do this, and it is good practice for you to do try to do this both ways.

Method I.
[As usual] Assume \( S \) is a nonempty set of real numbers which is bounded below. [We want to prove \( S \) has an infimum.]

Then, here’s the technique (you should be able to provide the precise proof).
• Consider the set \( \text{LB}(S) \).
• Show that this set \( \text{LB}(S) \) is nonempty and bounded above.
• Use the Completeness Axiom to conclude that \( \text{LB}(S) \) has a supremum. (obvious)
• Show that the supremum of \( \text{LB}(S) \) is the infimum of \( S \).

[Good exam topics: Prove the statements in the second and fourth bullets.]
Method II.

As usual] Assume $S$ is a nonempty set of real numbers which is bounded below. [We want to prove $S$ has an infimum.]

Then, here’s the technique (you should be able to provide the precise proof).

- Consider the set $-S$ defined as follows:
  
  $x$ is in $-S$ iff $-x$ is in $S$.

- (this is equivalent to $x$ is in $S$ iff $-x$ is in $-S$.

- Show that this set $-S$ is nonempty and bounded above.

- Use the Completeness Axiom to conclude that $-S$ has a supremum, $\sup(-S)$. (obvious)

- Show that the $-\sup(-S)$ if the infimum of $S$.
  (which can be written in two equivalent ways:
  $-\sup(-S) = \inf(S)$, of $\sup(-S) = -\inf(S)$).

[Good exam topics: Prove the statements in the second and fourth bullets.]

Reminder: In each case, Method I and Method II, we find a number which we want to prove is $\inf(S)$. What are the two facts we should prove to prove infimum?

What are the two ways to prove the second fact?
WOP discussed earlier.

Well-Ordering Property of the natural numbers:

Every nonempty subset of the natural numbers has a minimum.

What about the integers?

Obviously this is not true in the integers: The set of negative integers does not have a least element.

Why does it work in the natural numbers but not in the integers?:

Because the natural numbers themselves have a least element ...

Or, more generally, the natural numbers are bounded below ...

   by definition, the natural numbers are positive integers,
   so every natural number is greater than ...

This suggests the Well-Ordering Property for the integers:

   Every NONEMPTY subset of the integers which is bounded below has a minimum.

Which is often stated in the following form:

   Every nonempty subset of the integers which is bounded below has a least element.

This can be proved using the W.O.P. for the natural numbers.
Assume that \( S \) is a nonempty subset of \( \mathbb{Z} \) which is bounded below. Let \( b \) be a lower bound. (For all \( n \) in \( S \), \( n \geq b \).)

[Thinking: If we subtract \( b \) from every element of \( S \), we will get numbers which are \( \geq 0 \). If we subtract \( b \) and then add \( 1 \) to everyone element of \( S \), then we get numbers which are \( > 0 \); i.e., they’re natural numbers. That’s what we want.]

So we have a set \( S \) of integers with a lower bound \( b \).

We denote by \( S - b + 1 \) the set obtained by subtracting \( b \) and adding \( 1 \) to every element of \( S \). As just noted, \( S - b + 1 \) is a subset of the natural numbers. It is nonempty because \( S \) is nonempty (you should be able to prove this).

So, by W.O.P, \( S - b + 1 \) has a least element.

Use this least element of \( S - b + 1 \) to get a least element of the original \( S \).

Let \( m \) be the least element of \( S - b + 1 \).

Then \( \ldots \ldots \ldots \) is the least element of \( S \).

(You should be able to prove this.

Remember, you want to prove it’s the minimum of \( S \).

What two things do you have to prove?)