Two reasons for doing this:
  • Example of functions and, as we will see later, relations.
  • Important mathematical terminology (finite and infinite).

Think about different meanings of the size of a set:
  • How many things are in the set
  • How much space the set takes up.
Two different ideas of “size”.

The second idea is related to the idea of bounded that we have talked about.
But it doesn’t specifically refer to the number of elements in the set.

Now we focus on size in terms of the number of elements in a set (regardless of how much space the set takes up) and we start by thinking about counting:
For example,
How do you determine how many chairs are in a classroom?
What exactly are you doing when you count?
Suppose there are exactly 6 chairs in a classroom.

Think about two sets:
- The set of natural numbers, \( \{1, 2, 3, 4, 5, 6\} \),
- and the set of chairs in the classroom, call it \( C \).

Count.

\begin{align*}
\text{1} & \quad \text{2} & \quad \text{3} & \quad \text{4} & \quad \text{5} & \quad \text{6}
\end{align*}

Function.

Counting the chairs (one, two, three, ...) sets up a one-to-one function from the set \( \{1, 2, 3, 4, 5, 6\} \) onto the set \( C \) of chairs.
(Or, from the chairs onto the set of 6 numbers.)
Let $\mathbb{Z}_n$ denote the set of natural numbers $\leq n$:

$$\mathbb{Z}_n = \{ k \in \mathbb{N} : k \leq n \}.$$

**To say that the set $C$ has exactly 6 elements means that there is a bijection from $\mathbb{Z}_6$ to $C$.**

More generally, let $n$ be a natural number. We say that a set $S$ has exactly $n$ elements iff there exists a bijection $f : \mathbb{Z}_n \to S$.

More generally still, we talk about this idea between ANY two sets: We say that nonempty sets $A$ and $B$ are **equinumerous** iff there exists a bijection $f : A \to B$.

Language:

- $A$ and $B$ are equinumerous.
- $A$ is equinumerous with $B$.
- $A$ and $B$ have the same **cardinality**.

Note that this is defining a relation on the set of all sets (if there were such a thing). More exactly, it determines a relation on any nonempty collection of sets (this way, we don’t have to talk about “the set of all sets”).

This is the official, precise definition of that it means for sets $A$ and $B$ to have the same number (“equinumerous”) of elements. It describes the sets as being the same size, where size refers to the number of elements: A set of 100 ants is the same size as a set of 100 elephants.

As usual, we have to give special consideration to the empty set: We say that $\emptyset$ and $B$ are equinumerous iff $B = \emptyset$.

And, of course, in keeping with the terminology “the set $S$ has $n$ elements” (for a natural number $n$), we say that the empty set has 0 elements, and it is the only set with 0 elements.
Finite and infinite.

Suppose $S$ is a set. We say that $S$ is finite iff $S = \emptyset$ or there exists a natural number $n$ such that $S$ has exactly $n$ elements.

We say that $S$ is infinite iff $S$ is not finite

Note that “infinite” refers to the size of a set in terms of the “number of elements” of the set, not how much space the set occupies.

Unbounded and infinite refer to different aspects of the “size” of a set.
Any set of numbers which is unbounded (not bounded above or not bounded below) will be infinite.
But there are many bounded sets which are infinite.
Review/summary.
In this part of the course we focus more on ideas and definitions than on proofs, although there will be some proofs involving applications of previous ideas of functions and equivalence relations.

We say that nonempty sets $A$ and $B$ are **equinumerous** iff there exists a bijection $f : A \to B$.
Equivalent language:

$A$ and $B$ are equinumerous iff $A$ is equinumerous with $B$.

We extend this definition to the empty set by saying that a set $S$ and the empty set are equinumerous iff $S = \emptyset$.
I.e., a set $S$ is equinumerous with $\emptyset$ iff $S = \emptyset$.

Different notations are used for the “equinumerous” relation; we will use $\sim$:
For sets $A$ and $B$, $A \sim B$
iff $A$ and $B$ are equinumerous
iff there exists a bijection $f : A \to B$.
We consider this as a relation defined on any nonempty collection of sets.

We say that a set $S$ is **finite** iff

$S = \emptyset$ or there exists a natural number $n$ such that $S \sim \mathbb{Z}_n$,
where $\mathbb{Z}_n = \{ k \text{ in } \mathbb{N} : k \leq n \} = \{ 1, 2, ..., n \}$.

We say that a set $S$ is **infinite** iff $S$ is not finite.
It can be proved that, for very set $S$, if there exists $n$ as described above, then there exists only one such $n$.

In this case, we say that $S$ has $n$ elements, or, more precisely, $S$ has exactly $n$ elements. One notation for this (there are others) is $\#(S) = n$.

If $S = \emptyset$, then of course we say that $S$ has 0 elements. $\#(\emptyset) = 0$.

Everyone is familiar with examples of finite sets, and, for natural number $n$, sets with exactly $n$ elements.

For the record, without proof, here are examples of infinite sets which we have discussed in this class:

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$; any interval of more than one element --

\[ \text{e.g., for } a < b, \text{ the intervals } (a, b), (a, b], [a, b), \text{ and } [a, b]. \]

For an infinite set $S$, $\mathcal{P}(S)$, the power set of $S$, is infinite.

If $S$ is infinite, then the following sets are infinite:

- $S \cup T$, for any set $T$;
- $S \times T$ for any nonempty set $T$.

If $T$ is finite and $S \subseteq T$, then $S$ is finite.

Equivalently, if $S$ is infinite and $S \subseteq T$, then $T$ is infinite.

*End of April 26.*