Introduction to Finite and Infinite sets and Equinumerous

Refer to Textbook, Section 13.2, for “finite” and “infinite”

Reminder, sets can be divided up into “sizes” as follows:

<table>
<thead>
<tr>
<th>finite</th>
<th>infinite</th>
</tr>
</thead>
<tbody>
<tr>
<td>finite</td>
<td>denumerable</td>
</tr>
<tr>
<td></td>
<td>uncountable</td>
</tr>
<tr>
<td>countable</td>
<td>uncountable</td>
</tr>
</tbody>
</table>

_Corrected from April 28:_ Every set belongs to one of the categories in each row, and to only one in each row; for the top and bottom rows, this is by definition of “in”finite and “un”countable. in the middle row, “only one” requires more proof than we have given.

Examples: The set of students enrolled at UA is finite (“obvious”).
The sets \( \mathbb{N}, \mathbb{Z}, \) and \( \mathbb{Q} \) are countable (the first by definition).
The set \( \mathbb{R} \) is uncountable.

Suppose \( A \subseteq B \). Facts: Believable, but not proved here:

<table>
<thead>
<tr>
<th>A</th>
<th>( \subseteq )</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>( \Leftrightarrow )</td>
<td>F</td>
</tr>
<tr>
<td>D</td>
<td>( \Rightarrow )</td>
<td>D, U</td>
</tr>
<tr>
<td>F, D</td>
<td>( \Leftrightarrow )</td>
<td>D</td>
</tr>
<tr>
<td>U</td>
<td>( \Rightarrow )</td>
<td>U</td>
</tr>
<tr>
<td>F</td>
<td>( \Rightarrow )</td>
<td>F, D, U</td>
</tr>
<tr>
<td>F, D, U</td>
<td>( \Leftrightarrow )</td>
<td>U</td>
</tr>
</tbody>
</table>

As explained in class, the significance of the arrows here is that,
e.g., if the “larger” set \( B \) is finite, then the “smaller” set \( A \) is finite;
and, e.g., if the “smaller” set \( A \) is denumerable, then the “larger” set \( B \) is denumerable or uncountable.
FACT, with idea of proof:

The interval $\mathbb{\text{(0, 1)}}$ is not denumerable.

Proof by contradiction: Suppose $\mathbb{\text{(0, 1)}}$ is denumerable. Then, by definition of denumerable, $\mathbb{\mathbb{N}}$ and $\mathbb{\text{(0,1)}}$ are equinumerous, so, by definition of equinumerous, there exists bijection $f: \mathbb{\mathbb{N}} \to \mathbb{\text{(0, 1)}}$.

Let’s write $f(1) = x_1$, 
  $f(2) = x_2$, 
...
In general, $f(n) = x_n$.
We will get a contradiction. 
[Comment on writing general set $S = \{ x_1, x_2, \ldots, x_n, \ldots \}$ .]

Since $f$ is a bijection, it is surjective, so every element of $\mathbb{\text{(0, 1)}}$ is included in the outputs of $f$; for every element $a$ of $\mathbb{\text{(0, 1)}}$, there is an element $n$ of $\mathbb{\mathbb{N}}$ such that $a = x_n$.
(Also, since $f$ is injective, all the outputs are different; no two outputs with different subscripts are equal.)

We will get a contradiction by finding an element (which we will call $z$) in $\mathbb{\text{(0, 1)}}$ which is not in the list above.

We will use:
FACT (which should be proved, but which everyone “knows”):

Every real number has a decimal expansion; in particular, for every real number $x$ in $\mathbb{\text{(0, 1)}}$, we can write this as, there exist natural numbers $x_n$ with $0 \leq x_n \leq 9$ such that $x$ can be written $0.x_1x_2x_3x_4 \ldots$.
So let’s write our list of elements of \((0, 1)\) using decimals:

\[
\begin{align*}
f(1) &= x_1 = 0.x_{11}x_{12}x_{13}x_{14} \ldots \\
f(2) &= x_2 = 0.x_{21}x_{22}x_{23}x_{24} \ldots \\
&\quad \vdots \\
f(n) &= x_n = 0.x_{n1}x_{n2}x_{n3}x_{n4} \ldots \\
&\quad \vdots
\end{align*}
\]

Now we create/manufacture our number \(z\) different from all of these by giving it’s decimal expansion

\[
z = 0.z_1z_2z_3z_4 \ldots .
\]

We will use the digits 7 and 8 in the construction; many other pairs of digits could be used.

Look at \(x_1 = 0.x_{11}x_{12}x_{13}x_{14} \ldots \).

Look at \(x_{11}\). If \(x_{11} \neq 7\), let \(z_1 = 7\); if \(x_{11} = 7\), let \(z_1 = 8\).

So \(z_1 \neq x_{11}\).

So far, \(z\) looks like \(0.z_1\ldots\), and this can’t be \(x_1\).

Look at \(x_2 = 0.x_{21}x_{22}x_{23}x_{24} \ldots \).

Look at \(x_{22}\). If \(x_{22} \neq 7\), let \(z_2 = 7\); if \(x_{22} = 7\), let \(z_2 = 8\).

So \(z_2 \neq x_{22}\).

Now, \(z\) looks like \(0.z_1z_2\ldots\), and this can’t be \(x_1\) or \(x_2\).

Continue in this way; the pattern should be clear:

Look at \(x_n = 0.x_{n1}x_{n2}x_{n3}x_{n4} \ldots \).

Look at \(x_{nn}\). If \(x_{nn} \neq 7\), let \(z_n = 7\); if \(x_{nn} = 7\), let \(z_n = 8\).

So \(z_n \neq x_{nn}\).

Now, \(z\) looks like \(0.z_1z_2\ldots z_n\ldots\),

and this can’t be \(x_1\) or \(x_2\) or \(x_n\).
Continuing in this way, with $z = 0.z_1z_2... z_n...$, by choosing $z_n = 7$ if $x_{nn} \neq 7$, and $z_n = 8$ if $x_{nn} = 7$, we get a number $z$ which is not equal to any number in our list, because, for each, $z_n \neq x_{nn}$.

This is the contradiction; an element $z$ in $(0, 1)$ which is not in the range of our supposed bijection $f : \mathbb{N} \rightarrow (0, 1)$. Thus, there is no such bijection, and so $\mathbb{N}$ and $(0, 1)$ are not equinumerous.

Other ways of looking at what we proved:

- There is no surjective function $f : \mathbb{N} \rightarrow (0, 1)$, because what we essentially proved that such $f$ can’t be surjective; injective was not important.

- We can’t list the elements of $(0, 1)$ as a sequence $x_1, x_2, \ldots, x_n, \ldots$. 