Be sure to read the Function Notes under Announcements on the class website, and the material on functions in Ch. 11 of the textbook.

Read discussion of “three methods of proof”. (To be posted on website)

Additional comment on surjective: $f$ maps $A$ onto $B$.
If $f: A \to B$ is surjective, we say that $f$ maps $A$ onto $B$.

Here is an important example for thinking about aspects of functions:

We call it the “light function”, $L$: We have a classroom with students sitting in it and a whiteboard at the front of the room.

The set of all students is the domain of our function $L$. The set of points on the whiteboard is the codomain. Each student in the room has a light, like a flashlight, which he or she shines on the whiteboard. We imagine that the light shines on a single point on the board.

Then this gives us the function $L$. For each $x$ the classroom, $L(x)$ is the point on the whiteboard on which $x$ is shining the light.

If $C$ is the set of students in the classroom, and $W$ is the set of points on the whiteboard, then

$$L : C \to W.$$  

Questions for you to answer:

How can you picture the range of $L$?

Is $L : C \to W$ surjective? I.e., does $L$ map $C$ onto $W$?

Is $L$ injective? Or, better, under what circumstances is $L$ injective?
Now we will discuss the ideas of “image” and “pre-image” for subsets of the domain and codomain of a function. We will introduce the idea by means of this same example.

Suppose that some of the students turn off their lights, so we have only a subset, say $S$, shining lights on the board. (In general, $S$ could be empty, or $S$ could be the entire set $C$. For illustration, temporarily think of $S$ as a proper subset of $C$, $S$ is nonempty and $S \neq C$.)

The set of points in the codomain $W$ which are the outputs of the elements of $S$ is called the image of $S$, or the image of $S$ under $L$, and is denoted by $L(S)$.

So a point is in the image $L(S)$ iff it is an output from an input in $S$. I.e., $y$ is in $L(S)$ iff there is $x$ in $S$ such that $y = L(x)$.

In set builder notation, $L(S) = \{ y \}$.

Although we developed this idea for the specific example, this is the general definition of the image $L(S)$ for any function $L$ and any subset $S$ of the domain of $L$. To repeat: For any function $L$ and any subset $S$ of the domain of $L$,

$y$ is in $L(S)$ iff there is $x$ in $S$ such that $y = L(x)$.

CAUTION: Notation: Note that $L(S)$, where $S$ is a SUBSET of the domain, has a different meaning than $L(x)$, where $x$ is an ELEMENT of the domain.

You think about other examples, using your favorite function.
Now we will turn things around and go the other direction.  
(Still thinking first about our example of the light function.)

Let $T$ be a subset of the codomain $W$.  (ANY subset.)

Consider everyone in the domain (C) who is shining their light into the set $T$.

\[ \text{Domain} = C = \text{set of all students in classroom}. \]
\[ \text{Codomain} = W = \text{set of points on whiteboard}. \]
(The three students on the right in $C$ are shown shining their lights into $T$. Of course, the other students are also shining their lights into $W$, but not into $T$.)

The set of students who are shining their lights into $T$ is called the pre-image of $T$, or the pre-image of $T$ under $L$, denoted by $L^{-1}(T)$. In the example shown above, this is the set of 3 students on the right in $C$.

CAUTION. Notation. Note that this looks like “inverse function”. It is NOT inverse function.

**Don’t confuse it with inverse function.** We have not talked about inverse functions, and depending on time, we might not discuss them.

So, the pre-image $L^{-1}(T)$ consists of all the inputs which have an output in $T$.

I.e., $x$ is in $L^{-1}(T)$ iff $L(x)$ is in $T$.

In set-builder notation,
\[ L^{-1}(T) = \{ x : L(x) \text{ is an element of } T \} \]

Very very simple definition. This, and “reflexive”, will be the simplest definitions you get in this course.

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Again, the definition of pre-image developed for this specific function $L$ applies to any function $L$ and any subset $T$ of the codomain.

Learn it, and understand it.

We will rewrite it using our usual general notation $f : A \rightarrow B$

The pre-image of a subset $T$ of the codomain $B$ is the set of all inputs whose outputs are in $T$.

$x$ is in the pre-image $f^{-1}(T)$ iff $x$ is in the dom$(f)$ and $f(x)$ is in $T$.

$x \in f^{-1}(T)$ iff $[x \in \text{dom}(f)] \text{ and } f(x) \in T$

<Comment: We put the expression $[x \in \text{dom}(f)]$ in brackets because it is a “technicality”.

$x$ HAS to be in dom$(f)$ in order for $f(x)$ to make sense. But in the first round of learning this, learn it without “$x \in \text{dom}(f)$”, and then add this technicality later to make sure your definition makes sense.

Short form:

$x \in f^{-1}(T)$ iff and $f(x) \in T$.

Think about examples with numerical functions, such as quadratic functions or trig functions.
Summary of what we have done with functions:

Essential things to know about functions:

• What a function is, how to define a function,  
  (note the ambiguity in “define”)

• What the notation $f : A \rightarrow B$ means,

• Domain, codomain.

• Injective, surjective.

• Image, pre-image.