Functions  Injective and Surjective

Be sure to read the Function Notes under Announcements on the class website, and the material on functions in Ch. 11 of the textbook.

Properties of functions.
(Somewhat analogous to the properties “reflexive, symmetric, transitive” for relations.)

Two important properties: Injective and Surjective.

**Injective.** Recall the idea of “one-to-one”: Different inputs have different outputs. “Injective” is another word for “one-to-one”.

Definition. Suppose \( f : A \rightarrow B \).

We say that \( f \) is **injective** iff

for all \( a \) and \( b \) in \( \text{dom}(f) \), if \( a \neq b \), then \( f(a) \neq f(b) \).

Or: If \( a \) and \( b \) are in \( \text{dom}(f) \) and \( a \neq b \), then \( f(a) \neq f(b) \).

This says, precisely, that different inputs have different outputs.

**COMMON ERROR.** Don’t confuse “injective” with the function property. Often, when students are asked to describe one of these ideas, the words they use are actually describing the other idea.

What you should be able to do:

Know and understand the definition.

(Since “injective” is the same as “one-to-one”, you should already be familiar with the idea and be able to think of examples.)

Be able to identify functions as injective or not injective.

Be able to prove that a given function is injective, or be able to prove that a given function is not injective.
We will first discuss how to prove that a function is NOT injective.

Since the defining condition for injective is

\[ \text{for all } a \text{ and } b \text{ in } \text{dom}(f), \text{ if } a \neq b, \text{ then } f(a) \neq f(b), \]

usually the best and easiest way to prove that a function is not injective is to give a counterexample: Show there exists a and b in dom(f) such that a \neq b and f(a) = f(b).

In other words, find 2 different elements of the domain that have the same output.

Well-known example: Let’s take our old friend, the squaring function, SQ : \( \mathbb{R} \to \mathbb{R} \), given by,

\[ \text{for every real number } x, \text{ } SQ(x) = x^2. \]

Everyone can find two numbers; a and b, such that a \neq b and SQ(a) = SQ(b).
But, if you’re doing it, BE SPECIFIC. E.g., look at -2 and 2.
Each of these numbers is in the domain \( \mathbb{R} \) (usually, as here, this is so obvious it might not be mentioned, but BE CAREFUL).
Also, -2 \neq 2 and SQ(-2) = 4 = SQ(2).
So, SQ is not injective.

Now let’s look at proving that a function is injective. There is a standard way of doing this (not ALWAYS the way to do it, but the usual, standard way); this will be referred to as “the standard way” of proving that a function is injective.

The definition as we give it involves \( \neq \):

\[ \text{for all } a \text{ and } b \text{ in } \text{dom}(f), \text{ if } a \neq b, \text{ then } f(a) \neq f(b). \]

It is usually easier to deal with “=” rather than “\( \neq \)”, so let’s look at the contrapositive of the if-then statement here. We get

\[ \text{for all } a \text{ and } b \text{ in } \text{dom}(f), \text{ if } f(a) = f(b), \text{ then } a = b. \]

THIS is “the standard way” of proving that f is injective, use the statement just given.
(Note that this says, if the outputs are the same, then the inputs are the same.)
Example. Let’s look at the square root function, which we will denote by \( \text{SQRT} \).

\[
\text{SQRT} : [0, \infty) \rightarrow \mathbb{R}, \text{ given by, }
\]

for every nonnegative real number \( x \), \( \text{SQRT}(x) = \sqrt{x} \).

To prove injective, we want to prove for all \( a \) and \( b \) in \( \text{dom} (\text{SQRT}) \), if \( \text{SQRT}(a) = \text{SQRT}(b) \), then \( a = b \).

We start the proof of this universally quantified if-then statement in the usual way:

Proof.

Assume \( a \) and \( b \) in \( \text{dom} (\text{SQRT}) \) and \( \text{SQRT}(a) = \text{SQRT}(b) \).

So \( a \geq 0 \), \( b \geq 0 \), and \( \sqrt{a} = \sqrt{b} \).

Squaring both sides of the equality, we get \((\sqrt{a})^2 = (\sqrt{b})^2\), so \( a = b \).

We have shown that if \( \text{SQRT}(a) = \text{SQRT}(b) \), then \( a = b \), for all \( a \) and \( b \) in domain of \( \text{SQRT} \).

Thus, \( \text{SQRT} \) is injective.

Discussion of language (notation).

Note that we used the words and symbols

“We say that \( f \) is injective iff ... .”

We did not use the words and symbols

“We say that \( f(x) \) is injective iff ... .

The textbook uses the latter notation.

It is important to recognize the difference between these two notations.

What is the difference between \( f \) and \( f(x) \)?

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We have discussed the idea of an **injective** function. Note that if we have a function $f: A \rightarrow B$, then the set $B$ is irrelevant for “injective”. The requirement simply is

| if $x$ and $y$ are in $A$ (the domain of $f$), and if $f(x) = f(y)$, then $x = y$. |

Also, recall that we had two different ways of looking at “injective”: The condition above (if the outputs are the same, then the inputs are the same), and the contrapositive (if the inputs are different, then the outputs are different).

Now we introduce the idea of **surjective**. Again, we will have two ways of looking at this (but it has nothing to do with contrapositive in this case).

**Simple Definition:** We say that $f: A \rightarrow B$ is **surjective** iff

\[ \text{ran}(f) = B. \]

I.e., a function $f$ from a set $A$ to a set $B$ is surjective iff

$B$ is the range of $f$.

So, using this definition, if we want to prove that $f: A \rightarrow B$ is surjective, we have to prove that two sets are equal, a “two-way” proof.

But if we examine all the definitions more carefully, there is a simpler way of doing this proof, even though writing out the idea is more complicated.

*End of Mar 22*