Now, we talk about the PROOF of the PMI, based on another, apparently more fundamental, and certainly simpler, fact about the natural numbers.

First, some terminology, which we will talk about much more later.

Suppose $S$ is a set of real numbers.

We say that a number $m$ is the LEAST element of the set, or the SMALLEST element of the set, or the MINIMUM of the set, if and only if (iff)

- $m$ is an element of the set, and
- for all $x$ in $S$, $m \leq x$ [or $x \geq m$].

Think of examples ...

Think about: Does every subset of the real numbers have a minimum? Does every subset of the positive numbers have a minimum?

Basic property of natural numbers

Every subset of the natural numbers has a least element.
Revised basic property of natural numbers

Every NONEMPTY subset of the natural numbers has a least element.

This is called the Well Ordering Property (of the natural numbers).
We take it as an “axiom” of the natural numbers. It is very simple, we believe it, and we assume it’s true for the natural numbers.

Think about it, understand it, believe it, and -- sometimes most important -- realize that it does NOT apply to general sets of real numbers, even sets of positive numbers. It is a special property of the INTEGERS (the positive integers, although it can be extended a little beyond just positive integers).

Now we will use the Well Ordering Property to prove the Principle of Mathematical Induction
( use WOP to prove PMI).

It’s important to understand what this means, and to understand the difference between USING the PMI to prove something, and PROVING the PMI.
It’s sort of like USING a car to go somewhere, and BUILDING the car.
The PMI is a “tool” we use to prove things.
The car is a “tool” we use to go places.
We want to prove this:

The Principle of Mathematical Induction (simple, basic form):

Assume that for each natural number $n$, we are given a statement $P(n)$.

Assume that

(B) $P(1)$ is true, i.e., $P(n)$ is true for $n = 1$,

and

(I) for all $n \geq 1$, if $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \geq 1$.

(B) and (I) label the two assumptions, the “base case” assumption and the “inductive step” assumption.
Before we begin, a review of some fundamental ideas about strategies for proving things:

You have been advised that, as a general rule, to prove that a statement is FALSE, you prove that the negation is true. For universally quantified statements, or if-then statements, this is the same as finding a counterexample.

Two fundamental errors students make in this regard:

Fundamental logical error: You have NOT been advised that, as a general rule, to prove that an existential statement is FALSE, you find a counterexample. [Think about it.]

- You have NOT been advised that, as a general rule, to prove that a statement is FALSE, you prove that the negation is true.

- Fundamental strategic error: You have NOT been advised that, as a general rule, to prove that a statement is TRUE, you prove that the negation is FALSE.

Three reasons for the latter:
1. Error in formulating negation.
2. Error in proving negation.
3. Error in principle of giving simple, straightforward proofs if possible, not to make proofs more complicated than necessary. Very often, the direct straightforward proof that the original is true is simpler than, or at least no more complicated than, the proof of the negation.

Now, back to proof of PMI.
We want to prove this:

The Principle of Mathematical Induction (simple, basic form):

Assume that for each natural number $n$, we are given a statement $P(n)$.
Assume that

(B) $P(1)$ is true, i.e., $P(n)$ is true for $n = 1$,
and

(I) for all $n \geq 1$, if $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \geq 1$.

Be sure you understand the difference between the assumptions in this theorem, and the conclusion that you want to prove.

This is a case where we do NOT prove the result directly. Instead, in a sense, we prove that negation is false, by using “contradiction”. We assume that the conclusion is false, and reach a contradiction, so the conclusion must be true.
We want to prove this:

The Principle of Mathematical Induction (simple, basic form):

Assume that for each natural number $n$, we are given a statement $P(n)$.

Assume that

(B) $P(1)$ is true, i.e., $P(n)$ is true for $n = 1$, and

(I) for all $n \geq 1$, if $P(n)$ is true, then $P(n+1)$ is true.

Then $P(n)$ is true for all $n \geq 1$.

As pointed out previously, we will prove this is true “by contradiction”. After assuming all the hypotheses, we assume that the conclusion is false, and reach a contradiction, so the conclusion must be true.
Proof.
Assume that for each natural number \( n \), we are given a statement \( P(n) \).
Assume that

\begin{align*}
(B) & \quad P(1) \text{ is true, i.e., } P(n) \text{ is true for } n = 1, \\
(I) & \quad \text{for all } n \geq 1, \text{ if } P(n) \text{ is true, then } P(n+1) \text{ is true.}
\end{align*}

We want to prove: Then for all \( n \geq 1 \), \( P(n) \) is true.

As stated, we will prove this “by way of contradiction”.

[Keep in mind the “hint”: We’re going to use the Well Ordering Property to prove this.]
Assume the desired conclusion is false.
So, there exists \( n \geq 1 \) such that \( P(n) \) is false.

[Remember, in the context of induction discussions, we always assume that the letter \( n \) refers to a natural number (of course, unless otherwise state).]
Let \( F = \{ n \geq 1 : P(n) \text{ is false} \} \).
This is a set of natural numbers.
By our assumption “there exists \( n \geq 1 \) such that \( P(n) \) is false”, \( F \) is a nonempty set of natural numbers.

Refer back to that hint above:
\( F \) is a nonempty set of natural numbers. What can we conclude?
So \( F = \{ n \geq 1 : P(n) \text{ is false} \} \) is nonempty and thus, by WOP, \( F \) has a least element. Let’s call it \( m \).

We have reached this point simply by assuming \( F \) is nonempty and using WOP.

Remember what it means to say that \( m \) is the least element of \( F \):

\[ m \text{ is in } F \text{ and everything in } F \text{ is } \geq m. \]

Now let’s recall the assumptions we’re making in the PMI. We’re assuming

- (B) \( P(1) \) is true, i.e., \( P(n) \) is true for \( n = 1 \),

and

- (I) for all \( n \geq 1 \), if \( P(n) \) is true, then \( P(n+1) \) is true.

Let’s think more about \( m \). Can \( m = 1 \)? (Look at (B).)

So, since \( m \neq 1 \), we conclude that \( m > 1 \).

So, let’s play the “\( m - 1 \)” trick.

What can we say about \( m - 1 \)?

(Two things: What kind of number is \( m \)

and where is it relative to \( F \)?)

So what can we say about \( P(m-1) \)?

What can we conclude from assumption (I) about \( P(m) \)?

Where’s the problem?

Good problem or bad problem?
So, we assumed that $P(n)$ is false for some $n$.
We looked at the smallest such $n$, calling it $m$.
By it’s nature, $m$ is in $F$, so $P(m)$ is false.
But since $m$ is the smallest such number,

$P(m-1)$ must be true.

And our assumption(1)then implies that $P(m)$ is true.

Contradiction, as desired.

Thus there is no $n \geq 1$ such that $P(n)$ is false,
and therefore for all $n \geq 1$, $P(n)$ is true.

Done.