Partial Review from Jan 27

Example:
\[ \exists x \text{ in } \mathbb{R} \text{ such that } x + 3 = 1. \]

Proof. Choose \( x = -2 \).
Then \( x \) is a real number, and
\[ x + 3 = -2 + 3 = 1. \]
So,
\[ \exists x \text{ in } \mathbb{R} \text{ such that } x + 3 = 1. \]

Example:
\[ \exists x \text{ in } \mathbb{R} \text{ such that } x + 3 > 1. \]

The standard approach to such a proof (in simple cases like this) is to think about the statement; make a “guess” or “conjecture” at what a good \( x \) is; prove that it “works” in the statement.

Proof. Choose \( x = \_\_\_\_ \).
Then \( x \) is a real number, and
\[ x + 3 = \_\_\_\_ + 3 \_\_\_\_ \_\_\_\_\_\_\_. \]
So,
\[ \exists x \text{ in } \mathbb{R} \text{ such that } x + 3 > 1. \]
Example:
\[ \exists x \text{ in } \mathbb{R} \text{ such that } x^2 + 3 = \pi. \]

Proof. Choose \( x = \_\_\_\_\_\_\_\_\_ \).
Then \( x \) is a real number, because \( \pi - 3 > 0 \), and
\[ x^2 + 3 = \_\_\_\_\_\_\_\_\_ + 3 \_\_\_\_\_\_\_\_\_\_ . \]
So,
\[ \exists x \text{ in } \mathbb{R} \text{ such that } x^2 + 3 = \pi. \]

Negations of quantified statements.
\[ \forall x \text{ in } S, \ P(x) \]
\[ \exists x \text{ in } S \text{ such that } P(x) \]

\[ \neg [\forall x \text{ in } S, \ P(x)] \equiv \exists x \text{ in } S \text{ such that } \neg P(x) \]
\[ \neg \text{ is not distributive.} \]

Example:
For all \( x \) on UA football team, \( x \) weighs more than 230 lbs.

Negation:
There exists \( x \) on UA football team such that\( x \) does not weigh more than 230 lbs.

THINK ABOUT IT.
Similarly,

\[ \neg [\exists x \in S \text{ such that } P(x)] \equiv \forall x \in S, \neg P(x) \]

\[ \text{NOT } \equiv \forall x \text{ not in } S, \neg P(x) \]

Serious, serious, serious error.
\[ \neg \text{ is not distributive.} \]

Example:
There is an \( x \) in Math 323-2. such that \( x \) fails the class.

Negation:
For all \( x \) in Math 323-2, \( x \) does not fail the class.

Back to
\[ \neg [\forall x \in S, P(x)] \equiv \exists x \in S \text{ such that } \neg P(x) \]

A primary use of this equivalence is to answer the question, for a specific situation,
Is “\( \forall x \in S, P(x) \)” true or false?

A good way to prove that a statement is false is to prove that the negation is true.
So if we believe that
\( \forall x \in S, P(x) \)
is false, a good approach is to prove that the negation
\( \exists x \in S \text{ such that } \neg P(x), \)
is true.
Simple Example:
For every real number $x$, $x^2 > 0$.
This is false; we will prove the negation,
$\exists$ a real number $x$ such that $x^2 \leq 0$.

Be sure to understand that the last statement is the negation of the original statement.

Proof of negation. Choose $x = \ldots$. Then

Note what just happened. We wanted to prove that
For every real number $x$, $x^2 > 0$
is false. So, we proved the negation is true,
$\exists$ a real number $x$ such that $x^2 \leq 0$.

In order to do this, we found (exhibited) a real number $x$ which makes the predicate in the original statement, $x^2 > 0$, false.
Such an $x$ is called a counterexample for the original universally quantified statement.

In general, to prove that
$\forall x$ in $S$, $P(x)$
is false, the idea is to prove that the negation
$\exists x$ in $S$ such that $\neg P(x)$,
is true. And to prove that this existential statement, the negation of the original statement, is true, we find (exhibit) a $x$ in $S$ which makes the predicate in the original statement, $P(x)$, false.
Such an $x$ is called a counterexample.
A different kind of example, a direct proof of a statement with two quantifiers.

To prove:
\( \forall y \in \mathbb{R}, \exists x \in \mathbb{R} \) such that \( x - 3 > y \).

Proof. \([HODYSP?]\)

Assume \( y \in \mathbb{R} \).

[what next? Existential quest. We want to find \( x \) with a certain relation to \( y \). We want \( x - 3 > y \), so \( x > y + 3 \). Be specific: E.g., choose \( x = y + 4 \).]

Choose \( x = y + 4 \). Then \( x \in \mathbb{R} \) and
\[ x - 3 = (y + 4) - 3 = y + 1 > y. \]

So, \( \forall y \in \mathbb{R}, \exists x \in \mathbb{R} \) such that \( x - 3 > y \).

Note that exactly the same proof would work for

To prove:
If \( y \in \mathbb{R}, \exists x \in \mathbb{R} \) such that \( x - 3 > y \).
Another example.
To prove:
For all \( b > 0 \), there exists \( q > 0 \) such that
if \( x \in \mathbb{R} \) and \( 0 < x < q \), then \( 1/x > b \).

Proof. [HDYPSE?]
Assume \( b > 0 \).

[what next? Existential quest. We want to find \( q \) with a
certain relation to \( x \).
Note that if \( 0 < x < q \), then \( 1/x > 1/q \). So let’s choose \( q \) so
that \( 1/q = b \), so \( q = 1/b \).]
Choose \( q = 1/b \). This is possible, because \( b > 0 \),

[Now we check that \( q \) “works”.
if \( x \in \mathbb{R} \) and \( 0 < x < q \), then \( 1/x > b \).]
Suppose \( x \in \mathbb{R} \) and \( 0 < x < q \).
Then \( 1/x > 1/q \). Since \( 1/q = b \) by choice of \( q \), \( 1/x > b \).

For our choice of \( q \), we have proved: if \( x \in \mathbb{R} \) and \( 0 < x < q \),
then \( 1/x > b \).

So,
For all \( b > 0 \),
there exists \( q > 0 \) [namely, \( q = 1/b \)] such that
if \( x \in \mathbb{R} \) and \( 0 < x < q \), then \( 1/x > b \).

We have done a simple example (for exercise),
\[
\forall \ y \in \mathbb{R}, \ \exists \ x \in \mathbb{R} \ \text{such that} \ x - 3 > y.
\]
and a more complicated example (similar to something you might do
in Math 425)
For all \( b > 0 \), there exists \( q > 0 \) such that
if \( x \in \mathbb{R} \) and \( 0 < x < q \), then \( 1/x > b \).
You should study each of them, but not focus on memorizing the steps. Learn the ideas, and then if you have a problem which is, say, harder than the simple example but easier than the more complicated example, you can think about what to do to solve it.

Both examples were of the form ... $\forall \ y \ldots$, $\exists \ x$ such that ... . Notice that in each proof, we started with $y$, and then we found a specific $x$ which depended on $y$. Let’s look at an example with the quantifiers in the opposite order.

Consider the interval $(3, 4]$. To prove:

$\exists \ b \ \text{in} \ \mathbb{R} \ \text{such that} \ \forall \ x \ \text{in} \ (3, 4], \ b > x.$

(or $x < b$).

This time we START with an existential quest. We have to BEGIN by thinking, and we seek a number $b$ which is $>$ everything in $(3, 4]$. In this case, this is easy. It won’t always be so easy.
To prove:
\[ \exists b \in \mathbb{R} \text{ such that } \forall x \in (3, 4], \ b > x. \]  
(or \( x < b \)).

Proof. Choose \( b = 123 \).

[Now we want to prove \( \forall x \in (3, 4], \ b > x. \) HDSYP?]

Assume \( x \in (3, 4] \).

[We want to prove that our choice of \( b \) is \( > x \) (or \( x < b \)).]

Then (by definition of the interval \( (3, 4] \)), \( x \leq 4 < 123 \).

So, \( x < b \).

We have proved
\[ \exists b \in \mathbb{R} \text{ (namely, } b = 123 \) such that \( \forall x \in (3, 4], \ b > x. \]