Often in mathematics you have two sets, say $A$ and $B$, and you want to prove that the sets are equal.

A typical situation (but not the only situation) is when one of the sets, say $A$, is defined “abstractly”, in a general way, such as
- the range of a function;
- the null space of a linear transformation;
- an equivalence class of an equivalence relation (if you know what that is);
and the set $B$ is some specific set, such as a specific interval, or a specific set of vectors, or the set of all even numbers ...

Here is the general way to prove that two sets are equal:
you want to prove that $x$ is in $A$ iff $x$ is in $B$.

I.e.,
$$\forall x \in A, \ x \in B, \ \text{and} \ \forall x \in B, \ x \in A \ (A \subseteq B \ \text{and} \ B \subseteq A).$$

Usually better to do the “element-chasing” approach rather than the “subset” approach.

As usual, these can be stated as if-then statements:
If $x \in A$, then $x \in B$ and if $x \in B$, then $x \in A$.

So a proof usually has two “directions”: “going from $A$ to $B$”, and “going from $B$ to $A$”.

Common error: Doing only one direction. It will happen at least once this semester.

Sometimes it is more convenient to prove the contrapositive of at least one of these; e.g., prove
If $x \in A$, then $x \in B$ and if $x \notin A$, then $x \notin B$. 
As often happens, there are special considerations when one of the sets is given to be empty, e.g., when you have a set $S$ and you want to prove that $S = \emptyset$.

As we pointed out in last class, the statement

\[
\text{if } x \in \emptyset, \text{ then } x \in S
\]

is always true, for every set $S$, since “$x \in \emptyset$” is always false. Last time pointed out that this is equivalent to saying,

\[
\text{for every set } S, \emptyset \subseteq S
\]

(PLEASE don’t re-prove this everytime you use this fact; it’s true; if you need it, use it.)

In terms of proving, for a given set $S$, that $S = \emptyset$, we don’t have to “go both ways” as we usually do, because we know that one direction is always true.

So, roughly speaking, the idea is to start with an element, say $s$, of $S$, and prove that $s \in \emptyset$.

But even this is weird.
Here is the way to think about proving:

Given a set $S$, you want to prove.
We do a proof by contradiction.
Suppose $S$ is not empty, and consider $s \in S$.
Get a contradiction. Then $S = \emptyset$.

Summary: To prove that $S = \emptyset$, suppose $s$ is an element of $S$, and show that this leads to a contradiction.
Thus there is nothing in $S$, and $S = \emptyset$. 

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