Background and notation:

Archimedean Property:

\((\text{AP} >)\)  For every real number \(x\), there exists a natural number \(n\) such that \(n > x\).

To be proved equivalent to:

\((\text{AP} \geq)\)  For every real number \(x\), there exists a natural number \(n\) such that \(n \geq x\).

One of the purposes of this assignment (as emphasized in the assignment), was as an in “How Do You Start the Proof”, even though some of the proofs are “trivial”.

1(c) Prove that the Archimedean Property \((\text{AP} >)\) implies the Property \((\text{AP} \geq)\)

[The converse is 1(d) below.]
1(c) Prove that (AP >) implies (AP ≥) i.e., if (AP >) is true, then (AP ≥) is true.

**Proof.**

Assume (AP >).

[For every real number \( x \), there exists a natural number \( n \) such that \( n > x \).]

[To prove (AP ≥): For every real number \( x \), there exists a natural number \( n \) such that \( n ≥ x \).]

Assume \( x \) is a real number.

Then, by (AP >), there is a natural number \( n \) such that \( n > x \).

Since \( n > x \), then \( n ≥ x \)
(by definition of ≥).

This proves (AP ≥): For every real number \( x \), there exists a natural number \( n \) such that \( n ≥ x \).

See the next page for the converse, Problem 1(d).
1(d) Prove that \((AP \geq)\) implies \((AP >)\)
i.e., if \((AP \geq)\) is true, then \((AP >)\) is true.

Proof.

Assume \((AP \geq)\).

\[\text{For every real number } x, \text{ there exists a natural number } n \text{ such that } n \geq x.\]

To prove \((AP >)\):

\[\text{For every real number } x, \text{ there exists a natural number } n \text{ such that } n > x.\]

Assume \(x\) is a real number.

Then, by \((AP \geq)\), there is a natural number \(n\) such that \(n \geq x\).

Since \(n + 1 > n\), then \(n + 1 > x\).

This proves \((AP \geq)\): For every real number \(x\), there exists a natural number greater than \(x\).

Another approach directly below; see next page for best approach:

Assume \(x\) is a real number.

Then, by \((AP \geq)\), there is a natural number \(m\) such that \(m \geq x\).

Let \(n = m + 1\), so \(n > m \geq x\), so \(n > x\).

This proves \((AP \geq)\): For every real number \(x\), there exists a natural number \(n\) such that \(n > x\).
Recall that in the first part of this assignment, you are asked to write the AP in different ways, using different letters and without using letters. In the last proof, we used $m$ instead of $n$ for the natural number we are assuming exists.

Here is another approach, applying our assumption to a different "$x$".

1(d) Prove that $\text{(AP} \geq \text{)}$ implies $\text{(AP} > \text{)}$
i.e., if $\text{(AP} \geq \text{)}$ is true, then $\text{(AP} > \text{)}$ is true.

Proof.
Assume $\text{(AP} \geq \text{)}$.

[For every real number $x$, there exists a natural number $n$ such that $n \geq x$.]

[To prove $\text{(AP} > \text{)}$:
For every real number $x$, there exists a natural number $n$ such that $n > x$.]

Assume $x$ is a real number.

Then, by $\text{(AP} \geq \text{)}$, there is a natural number $n$ such that $n \geq x + 1$.

Since $n \geq x + 1 > x$.

This proves $\text{(AP} \geq \text{)}$: For every real number $x$, there exists a natural number $n$ such that $n > x$.

This is the simplest proof of the three, and it illustrates an important point. We know a statement $\text{(AP} \geq \text{)}$ is true for EVERY real number. We start our proof of $\text{(AP} > \text{)}$ with an arbitrary $x$, BUT WE DON’T HAVE TO USE THE ASSUMPTION $\text{(AP} \geq \text{)}$ FOR THIS SAME $x$. This is a very important point. We know something is true for every $x$, so we choose the $x$ to be what is convenient for us.