We will say that a set \( S \) is a **reflective set** iff \( S \) is a nonempty set of real numbers (i.e., a nonempty subset of \( \mathbb{R} \)) and for all \( x \) in \( S \), \(-x\) is in \( S \).

[Note that the word is **reflective**, which is not the same as **reflexive**. The word **reflective** is invented just for this problem.]

a. Suppose you want to prove that a nonempty subset \( S \) of the real numbers is reflective, using the definition of reflective.

What is the standard way of starting the proof that it is reflective, using the definition of reflective?

**SOLUTION.** The definition of “\( S \) is reflective” can be written either as

\[
\text{\( S \) is a nonempty set of real numbers and for all \( x \) in \( S \), \(-x\) is in \( S \).}
\]

or

\[
\text{\( S \) is a nonempty set of real numbers and if \( x \) in \( S \), then \(-x\) is in \( S \).}
\]

Since it is assumed that the given set \( S \) is a nonempty set of real numbers, we would start the proof that it is reflective in the usual way: **Assume \( x \) is in \( S \).**

b. True or false; prove your answer:

If \( R \) and \( S \) are nonempty sets of real numbers, \( R \) is reflective, and \( R \subseteq S \), then \( S \) is reflective.

**SOLUTION.** FALSE. Counterexample.

Let \( R = \{0\} \). Then \( R \) is a nonempty set of real numbers, and \( R \) is reflective.

\[
\text{Proof that \( R \) is reflective. Suppose \( x \in R \). Then, by definition of \( R \), \( x = 0 \).
\]

so \(-x = 0 \in R\), and thus \( R \) is reflective.

Let \( S = \{0, \pi\} \). Clearly, \( S \) is a nonempty set of real numbers, and \( R \subseteq S \). But \( S \) is not reflective.

\[
\text{Proof that \( S \) is not reflective. Consider \( x = \pi \). Then \(-\pi \in S \). So \( S \) is not reflective.}
\]

by counterexample.

We have found \( R \) and \( S \) such that

\[
\text{\( R \) and \( S \) are nonempty sets of real numbers, \( R \) is reflective, and \( R \subseteq S \).
}\]

But \( S \) is not reflective.

This proves that the statement in (b) is false.

**COMMENT.** If you believe that the statement given in (b) is true, a proof should start (of course) in the usual way for proving such statement.

First, assume \( R \) and \( S \) are nonempty sets of real numbers, \( R \) is reflective, and \( R \subseteq S \).

Then, to prove \( S \) is reflective, again, start in the usual way, using the definition of reflective:

**Assume \( x \in S \).**

**If you want to prove that \( S \) is reflective, you start the proof with an arbitrary element of \( S \), just as you should have said in part (a) of this problem. DON’T start with an element of \( R \), even though this is a certain kind of element of \( S \). Start with an ARBITRARY element of \( S \).**

That’s the point of all the talk about HDYSP.

Now, having started with an arbitrary element of \( S \), **try to prove \(-x \in S \). Can’t.**

So, find counterexample..
There were two homework problems similar to Problem 2.

Briefly, the similar homework problems were:

If \( R \) is a reflexive relation on a set \( A \) and \( S \) is a relation on \( A \) and \( R \subseteq S \), then \( S \) is reflexive on \( A \).

If \( R \) and \( S \) are relations and \( R \) is symmetric and \( R \subseteq S \), then \( S \) is symmetric.

The “reflective” problem on the exam is most similar to the “symmetric” problem for relations:

The definition of “\( S \) is symmetric” looks like:

For every \( (a, b) \) in \( S \), ... something happens ... .

(Completely: For every \( (a, b) \) in \( S \), \( (b, a) \) is in \( S \).)

The definition of “\( S \) is reflective” looks like:

For every \( x \) in \( S \), ... something happens ... .

(Completely: For every \( x \) in \( S \), \( -x \) in \( S \).)

To prove that a relation \( S \) is symmetric using the definition, one STARTS with an ordered pair \( (a, b) \) in \( S \) (not with an ordered pair \( (a, b) \) from somewhere else) ...

one starts with an ordered pair \( (a, b) \) in \( S \) and then shows \( (b, a) \) is in \( S \),

not start with \( (a, b) \) from somewhere and then show \( (a, b) \) and \( (b, a) \) are in \( S \).)

To prove that a set \( S \) is reflective using the definition, one STARTS with an element \( x \) of \( S \) (not with an element \( x \) from somewhere else) ...

one starts with an element \( x \) in \( S \) and then shows \( -x \) is in \( S \),

not with an element \( x \) from somewhere else and then show \( x \) and \( -x \) are in \( S \).