4. For the purposes of this problem, we say that a nonempty set \( S \) is **expansive** iff for all \( x \) in \( S \), there exists \( y \) in \( S \) such that \( y > x \).

(a) **True or False**: A nonempty set is expansive if it is unbounded above. Prove your answer.

(b) **True or False**: If a nonempty set is expansive, then it is unbounded above. Prove your answer.

**SOLUTION (a)**. This is true.

Proof.
Suppose \( S \) is a set which is unbounded above. We want to prove that \( S \) is expansive.

Reminder: Since \( S \) is unbounded above, then by the last part of Problem 3, for every real \( x \), there is \( y \in S \) such that \( y > x \). Now we prove that \( S \) is expansive.
Assume \( x \in S \). Since \( S \) is unbounded above, there is \( y \in S \) such that \( y > x \), as just stated.
This proves, by definition, than \( S \) is expansive.

**SOLUTION (b)**. This is false. Counterexample:
We will give a set which is expansive but not bounded above.
Let \( S \) be your favorite open interval -- it’s best to be specific, so let \( S = (0, 1) \).
We show that this is expansive:
Suppose \( x \in S \). Then \( 0 < x < 1 \). We know there is a real number \( y \) with \( x < y < 1 \), as proved (supposedly) in the first homework assignment of the semester. (E.g., \( y = (x + 1)/2 \).)
So \( y \) satisfies \( 0 < x < y < 1 \), and thus \( y \in S \) and \( y > x \).
We have shown that for every \( x \in S \), there is \( y \in S \) such that \( y > x \).
Thus, by definition, \( S \) is expansive.

**COMMENT.**

i. As mentioned frequently during the semester, if you have two distinct real numbers, you can find a number between them, and use can use this fact without reproving it every time you use it. In particular, if \( a < b \) and you use \( c = (a + b)/2 \), you shouldn’t repeat the prove that \( a < c < b \). This is a simple and outstanding example of “UPR”.

ii. The fact that nonempty bounded open intervals are “expansive”, as defined here, is a fundamental numerical fact of life in the real numbers (and in the rational numbers), and it is intimately connected to Problem on Project I. (Open intervals are both expansive “on the right”, as defined here, and “on the left”, with a similar definition.)