Let $S = \mathbb{Z} \times \mathbb{N}$. Define a relation $\equiv$ on $S$ by

$$(n, m) \equiv (p, q) \iff nq = mp.$$  

Prove this is an equivalence relation on $S$.

Think about what $S$ is; think about what the relation “looks like”, and THINK OF SOME EXAMPLES.

Proof. Reflexive.

Recall how to start even/odd proof.

Complete: HDYSP? Suppose $x \in S$.

We want to prove $x \equiv x$.

Then there exist $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ such that $x = (a, b)$.

Think about role of the $a$ and $b$.

We want to prove $x \equiv x$; i.e., $(a, b) \equiv (a, b)$; i.e., $ab = ba$.

...
Symmetric.

Complete: Assume \( x \equiv y \). We want to prove that \( y \equiv x \).

*Introduce ordered pairs as above.*

Concise: Assume \((a, b) \equiv (c, d)\).

\[ \text{We want to prove} \ (c, d) \equiv (a, b); \ i.e., \ cb = da. \]

By definition of the relation \( \equiv \), \( ad = bc \).

By the commutative law for the multiplication of integers, \( da = cb \), so \( cb = da \).

Thus, by definition of the relation, \((c, d) \equiv (a, b)\).

We have proved that if \((a, b) \equiv (c, d)\), then \((c, d) \equiv (a, b)\).

Thus, the relation is symmetric.

Transitive. Assume \((a, b) \equiv (c, d)\) and \((c, d) \equiv (p, q)\).

\[ \text{We want to prove} \ (a, b) \equiv (p, q); \ i.e., \ aq = bp. \]

By definition of the relation \( \equiv \), \( ad = bc \) and \( cq = dp \).

Then \( adcq = bcdp \).

\[ \text{So} \ aq = bp, \ \text{and thus} \ (a, b) \equiv (p, q). \]

Try again
Transitive. Assume \((a, b) \equiv (c, d)\) and \((c, d) \equiv (p, q)\).

We want to prove \((a, b) \equiv (p, q)\); i.e., \(aq = bp\).

By definition of the relation \(\equiv\), \(ad = bc\) and \(cq = dp\).
We have \(ad = bc\), so \(adq = bcq = bdp\),
so \(aqd = bpd\),
Since \(d \neq 0\), we can conclude that \(aq = bp\), so \((a, b) \equiv (p, q)\).
Therefore the relation is transitive.