This is a summary of the proof which was done in class on Wednesday, October 12.

Suppose \( f \) is a monotone function whose domain, \( D \), and range are both intervals, as described in Theorem 3.23 in the textbook. Here is part of the proof that \( f \) is continuous on \( D \) in the case \( f \) is increasing on \( D \).

Proof. Consider an element \( a \) of \( D \) and \( \varepsilon > 0 \). Using the \( \varepsilon-\delta \) definition of continuity, we want to find \( \delta \) such that

for all \( x \) in \( D \), if \( a - \delta < x < a + \delta \), then \( f(a) - \varepsilon < f(x) < f(a) + \varepsilon \). \hspace{1cm} (1)

We will first find \( \delta \) such that

for all \( x \) in \( D \), if \( a - \delta < x < a + \delta \), then \( f(x) < f(a) + \varepsilon \). \hspace{1cm} (2)

A similar proof will show that we can find \( \delta \) such that for all \( x \) in \( D \), if \( a - \delta < x < a + \delta \), then \( f(x) \geq f(a) - \varepsilon \).

Then, choosing our “final” \( \delta \) to be the minimum of these two, we have found \( \delta > 0 \) which has the desired property.

If for every \( x \) in \( D \), \( f(x) \leq f(a) \), then any \( \delta > 0 \) will work to satisfy (2).

So, suppose there is an \( x_1 \) in \( D \) such that \( f(x_1) > f(a) \). By monotonicity, \( x_1 > a \).

Let \( y_\varepsilon = \min\{ f(a) + \varepsilon/2, f(x_1) \} \). Since \( f(a) < y_\varepsilon \leq f(x_1) \) (be sure you understand why) and \( f(D) \) is an interval, \( y_\varepsilon \) is an element of \( f(D) \) so we can find \( x_\varepsilon \) in \( D \) such that \( f(x_\varepsilon) = y_\varepsilon \).

By the monotonicity of \( f \), \( x_\varepsilon > a \). (why?).

Let \( \delta = x_\varepsilon - a \). Suppose \( x \) is in \( D \) and \( a - \delta < x < a + \delta \),

As noted above, for \( x \leq a \), \( f(x) \leq f(a) < f(a) + \varepsilon \).

For \( a < x < a + \delta \), we have \( f(x) \leq f(a + \delta) = f(x_\varepsilon) = y_\varepsilon \leq f(a) + \varepsilon/2 < f(a) + \varepsilon \).

Thus, condition (2) is satisfied for this choice of \( \delta \).