Proof that mixed partial derivatives are equal.

Think about the following approximations for small $h$ and $k$.

$$D_1f(x) = h^{-1}[f(x + he_1) - f(x)] = \Delta_1, h f(x) \quad \text{think of } h \text{ as fixed; definition of } \Delta_1, h f \text{ applies to any } f.$$

$$D_2f(x) = k^{-1}[f(x + ke_2) - f(x)] = \Delta_2, k f(x) \quad \text{think of } k \text{ as fixed; definition of } \Delta_2, k f \text{ applies to any } f.$$

$$D_1D_2 f(x) = D_1 \Delta_2, k f(x) = \Delta_1, h \Delta_2, k f(x) = h^{-1}[\Delta_2, k f(x + he_1) - \Delta_2, k f(x)]$$

**COMPARE THESE TWO, LINE ABOVE AND LINE BELOW.**

$$D_2D_1 f(x) = D_2 \Delta_1, h f(x) = \Delta_2, k \Delta_1, h f(x) = k^{-1}[\Delta_1, h f(x + ke_2) - \Delta_1, h f(x)]$$

One sees that $\Delta_1, h \Delta_2, k f(x)$ and $\Delta_2, k \Delta_1, h f(x)$ are equal, and each represents, in a different way, an approximation to a second order partial derivative; first, $D_1D_2 f$, and second $D_2D_1 f$. We can use this to prove, with appropriate assumptions, the equality of these second-order partial derivatives.

$$D_1D_2 f(x) \quad [\text{this initial approximation is just for motivation of the following exact calculations}]$$

$$= \Delta_1, h \Delta_2, k f(x)$$

$$= h^{-1}[\Delta_2, k f(x + he_1) - \Delta_2, k f(x)] \quad \text{think of } k \text{ as fixed}$$

$$= h^{-1}[D_1 \Delta_2, k f(x + \theta_1 he_1) h] \quad \text{by MVT for } \Delta_2, k f$$

$$= D_1 \Delta_2, k f(x + \theta_1 he_1) \quad \text{note } \Delta_2, k f(x + \theta_1 he_1) = k^{-1}[f(x + \theta_1 he_1 + ke_2) - f(x + \theta_1 he_1)]$$

$$= k^{-1}[D_1 f(x + \theta_1 he_1 + ke_2) - D_1 f(x + \theta_1 he_1)]$$

$$= k^{-1}[D_2D_1 f(x + \theta_1 he_1 + \theta_2 ke_2) k]$$

$$= D_2D_1 f(x + \theta_1 he_1 + \theta_2 ke_2)$$

Similarly,

$$D_2D_1 f(x)$$

$$= \Delta_2, k \Delta_1, h f(x) \quad \text{this is equal to the expression circled above}$$

$$= k^{-1}[\Delta_1, h f(x + ke_2) - \Delta_1, h f(x)]$$

$$= D_1D_2 f(x + \theta_1', he_1 + \theta_2' ke_2)$$

So, there exist $z_1$ and $z_2$ arbitrarily close to $x$ such that $D_2D_1 f(z_1) = D_1D_2 f(z_2)$.

Thus, if the second order partial derivatives are continuous at $x$, $D_2D_1 f(x) = D_1D_2 f(x)$!