Let $S$ be a set. An $n$-dimensional coordinate system on $S$ is a one-to-one function mapping $S$ onto an open subset of $\mathbb{R}^n$. If we denote such a function by $x : S \to U \subseteq \mathbb{R}^n$, then we can write $x = (x^1, x^2, \ldots, x^n)$, where for each $i$, $x^i$ is a function from $S$ into $\mathbb{R}$.

Let $f$ be a function from $S$ into $\mathbb{R}^n$. We say that $f$ is $C^\infty$ iff the composition $f \circ x^{-1} : U \to \mathbb{R}^n$ is $C^\infty$. We denote by $\Phi$ the set of all REAL-valued $C^\infty$ functions on $S$. Given such a function $f$ and a coordinate system $x$ on $S$, we define, for each point $p$ in $S$, the partial derivatives of $f$ with respect to the coordinate functions $x^i$:

$$\frac{\partial}{\partial x^i} f(p) = \partial_i (f \circ x^{-1})(x(p))$$

This derivative is also denoted by $\frac{\partial}{\partial x^i}_p f$.

**NOTE.** In more general situations, the set $\Phi$ is taken to be the set of all real-valued $C^\infty$ functions defined in a neighborhood of $p$ (or defined near $p$). Thus $\Phi$ is specifically associated with the point $p$, just as the partial derivative operators are associated with $p$.

**EXERCISE 1.** Show that the coordinate system $x$ is a $C^\infty$ function on $S$ and that for each $i$, $x^i$ is in $\Phi$.

Fix a point $p$ in $S$. It is easy to show that $\frac{\partial}{\partial x^i}_p$ is a linear operator on the vector space $\Phi$ of all $C^\infty$ functions on $S$ and that this operator has the following property: For all $f$ and $g$ in $\Phi$,

$$\frac{\partial}{\partial x^i}_p (f g) = f(p) \frac{\partial}{\partial x^i}_p g + g(p) \frac{\partial}{\partial x^i}_p f$$

An operator $v$ on $\Phi$ is called a **linear derivation** (or just a derivation) on $\Phi$ at $p$ if $v$ is linear and $v(fg) = f(p)v(g) + g(p)v(f)$ for all $f$ and $g$ in $\Phi$.

**EXERCISE 2.** Verify that the operator $\frac{\partial}{\partial x^i}_p$ is a linear derivation on $\Phi$ at $p$.

**EXERCISE 3.** Verify that $\frac{\partial}{\partial x^i}_p (x^k) = \delta_{ik}$, where $\delta_{ik} = 0$ if $i \neq k$ and $\delta_{ik} = 1$ if $i = k$.

**EXERCISE 4.** Let $u = (u^1, u^2, \ldots, u^n)$ be an element of $U = \text{range}(x)$. Show that $x^i(x^{-1}(u)) = u^i$. 


We also consider \( C^\infty \) parametrized paths, without (for convenience) making the “smoothness” assumption that the derivative is nonzero. Let \( I \) be a nontrivial \textbf{closed bounded} interval in \( \mathbb{R} \). We say that a one-to-one function \( \gamma : I \to S \) is a \( C^\infty \) \textbf{parametrized path} in \( S \) iff the composition \( x \circ \gamma : I \to \mathbb{R}^n \) is continuous of \( I \) and is \( C^\infty \) on the interior of \( I \).

**EXERCISE 5.** Show that \( \gamma : (a, b) \to S \) is a \( C^\infty \) parametrized path in \( S \) iff for each \( f \) in \( \Phi \), the composition \( f \circ \gamma \) is a \( C^\infty \) real-valued function on \( (a, b) \). [This needs to be cleaned up to be consistent with definition above.]

Suppose \( \gamma \) is a \( C^\infty \) parametrized path such that \( \gamma(t_0) = p \), where \( t_0 \) is in the interior of the domain \( I \) of \( \gamma \). (So \( \gamma \) is a \( C^\infty \) parametrized path \textbf{through} \( p \).) Then the derivative \( (f \circ \gamma)'(t_0) \) exists; this defines an operator \( D\gamma \) on \( \Phi \):

\[
D\gamma f = (f \circ \gamma)'(t_0).
\]

Think of \( D\gamma f \) as the directional derivative of \( f \) in the direction of the tangent vector to \( \gamma \) at the point \( p \). (Of course, this operator depends on the point \( p \), but since we are considering \( p \) fixed, we will at least temporarily ignore this in the notation \( D\gamma \).)

**EXERCISE 6.** Show that \( D\gamma \) is a linear derivation on \( \Phi \) at \( p \).

**EXERCISE 7.** For each index \( i, 1 \leq i \leq n \), find a \( C^\infty \) parametrized path \( \gamma^{(i)} \) through \( p \) such that

\[
D\gamma^{(i)} = \left. \frac{\partial}{\partial x_i} \right|_p.
\]

(Thus, as in \( \mathbb{R}^n \), partial derivatives are special cases of directional derivatives.)

**EXERCISE 8.** Show that (for \( \gamma \) as above, with \( \gamma(t_0) = p \))

\[
D\gamma = \sum_{i=1}^n (x^i \circ \gamma)'(t_0) \left. \frac{\partial}{\partial x_i} \right|_p
\]

**EXERCISE 9.** Show that, given \( u \) in \( \mathbb{R}^n \) and \( p \) in \( S \), there exists a \( C^\infty \) parametrized path \( \gamma \) in \( S \) through \( p \) such that \( D\gamma = \sum_{i=1}^n u^i \left. \frac{\partial}{\partial x^i} \right|_p \).

**DEFINITION.** We call \( D\gamma \) a \textbf{tangent vector} to \( S \) at \( p \). The collection of all such tangent vectors is called the \textbf{tangent space} to \( S \) at \( p \) and denoted by \( T_p(S) \).

**THEOREM.** \( T_p(S) \) is an \( n \)-dimensional vector space. The set of partial derivatives \( \left. \frac{\partial}{\partial x_i} \right|_p \) is a basis for \( T_p(S) \).

**EXERCISE 10.** Verify this Theorem. (Most of it follows from the preceding exercises.)