On the Take-Home part of Exam 1, you were asked to discuss the correspondence between a linear functional $f$ on $\mathbb{R}^n$ and a vector $v$ in $\mathbb{R}^n$ given by $f(u) = \langle v, u \rangle$ (or, by symmetry, $f(u) = \langle u, v \rangle$) for all $u$ in $\mathbb{R}^n$. It is not difficult to see that this is a one-to-one correspondence, and, in fact, from an algebraic point of view, it is an isomorphism. We now look at a similar situation for bilinear functions.

A bilinear function on $\mathbb{R}^n \times \mathbb{R}^n$ is a real-valued function $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is linear in each variable separately; i.e., for each $u$ in $\mathbb{R}^n$, $\beta(u, v)$ is a linear function of $v$, and for each $v$ in $\mathbb{R}^n$, $\beta(u, v)$ is a linear function of $u$. A bilinear function $\beta$ is called symmetric iff $\beta(u, v) = \beta(v, u)$ for all $u$ and $v$. (We will be applying these results to derivatives only in the symmetric case, so you can assume symmetric if really helpful.)

1. The goal of this problem and the next two is to describe what bilinear real-valued functions (sometimes called bilinear functionals) on $\mathbb{R}^n \times \mathbb{R}^n$ “look like”.

(a) Given an $n \times n$ matrix $B$, define $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $\beta(u, v) = \langle Bu, v \rangle$ for each $u$ and $v$ in $\mathbb{R}^n$. Explain, in terms of the function $\beta$, the matrix $B$, and the scalar product, what it means for $\beta$ to be bilinear.
Just write down a few equations; you don’t have to prove anything beyond writing down the equations which show how properties of the scalar product imply the bilinearity of $B$.

(b) Suppose that the function $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is bilinear. Prove that there exists an $n \times n$ matrix $B = (b_{ij})$ such that $\beta(u, v) = \langle Bu, v \rangle$ for each $u$ and $v$ in $\mathbb{R}^n$.

2. Explain the relationship between a bilinear function $\beta$ and the idea of a quadratic function as defined at the bottom of page 380 in the textbook. This is very simple; it just requires sorting out the definitions.

3. Given a linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ (sorry for the $T$; some form of $B$ would be nicer, but it would make it hard to distinguish things in handwriting), define $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $\beta(u, v) = \langle Tu, v \rangle$ for each $u$ and $v$ in $\mathbb{R}^n$. Explain, in terms of the function $\beta$, the linear function $T$, and the scalar product, what it means for $\beta$ to be bilinear.
As in Problem 1, just write down one or two equations; you don’t have to prove anything beyond writing down the equations which show how properties of the scalar product and linear functions imply the bilinearity of $\beta$.
Describe the relation between the linear function $T$ here and the matrix $B$ in Problem 1 above.

(b) Suppose that the function $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is bilinear. Prove that there exists a linear function $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $\beta(u, v) = \langle Tu, v \rangle$ for each $u$ and $v$ in $\mathbb{R}^n$. 