4. Reminder of general background: Given a subset $A$ of $\mathbb{R}^n$, we have previously defined the derivative of a function $F : A \rightarrow \mathbb{R}^m$ at a point $x$ as a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$ (with certain properties, if it exists).

If $f$ is a real-valued function on a subset of $\mathbb{R}^n$ and $A$ is the set on which the gradient $\nabla f$ of $f$ exists, we have a function $\nabla f : A \rightarrow \mathbb{R}^n$; at each $x$ at which the gradient exists, $\nabla f(x)$ is an element of $\mathbb{R}^n$. Since the gradient $\nabla f$ is a function from a subset of $\mathbb{R}^n$ into $\mathbb{R}^n$, we can ask about its derivative. This (where it exists) will be a linear function (linear transformation) from $\mathbb{R}^n$ to $\mathbb{R}^n$, which we think of as a kind of second derivative of the original $f$ at the points for which it exists.

(a) Read the preceding paragraph again to make sure you know what is going on.

(b) The derivative of the vector-valued function $\nabla f$ can be denoted, naturally but somewhat unusually, by $(\nabla f)'(x)$ at any point $x$ where it exists. This is then a linear function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Can you guess what the matrix of this linear transformation is? (No proof necessary at this point; refer to page 381 of the textbook for more background.) Don’t just answer “yes” or “no” – if you have a guess, give it.

5. The following observation may be useful in Problem 6 below. Show that a symmetric bilinear function is completely determined by its quadratic function:

Let $\beta$ be a symmetric bilinear function and define its quadratic function $Q$ as suggested in Problem 2 above. By expanding $\beta(u + v, u + v)$, show that if you know the function $Q$, then you can “reconstruct” the function $\beta$.

to be continued on page 3