We have now discussed, in general, bilinear functions and the connection with linear transformations and matrices. We have also discussed the gradient of a real-valued function on $\mathbb{R}^n$ (as in the textbook) and the notion of the derivative of such a function as a linear transformation (see Notes on Derivatives).

6. Tie it all together, as described in parts (a) and (b) below, making liberal assumptions about existence and continuity of second order partial derivatives wherever convenient (as in Theorem 14.2) to give a reasonable definition of the second derivative $f''(x)$ of $f$ as a symmetric bilinear function $\beta: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. (Note: In the approach discussed here, the second derivative will not be the derivative of the first derivative, since that would involve somewhat more advanced concepts. But the bilinear function to be defined – by you – is very closely related to what would be the derivative of the first derivative.)

There are two ways to do this:

(a) Explicitly, in somewhat the same way as the textbook defines the gradient vector (except you will be defining a function, not a vector). This was hinted at in Problem 4 above. Using the notation and terminology of Theorem 14.2 (in Section 14.1) and Theorem 14.21 (in Section 14.3) in the textbook, as well as the preceding discussion of bilinear functions, define the second derivative $f''(x)$ as a symmetric bilinear function explicitly in terms of $(\nabla f)'(x)$ and in also terms of the Hessian matrix. You also might want to preview part (b) below before doing this; it all ties together.

(b) Implicitly, in a way analogous to the way we defined the first derivative, using an equation such as

$$f(x + h) = f(x) + f'(x)(h) + \beta(h, h) + R_f(x, h),$$

or

$$f(x + h) = f(x) + f'(x)(h) + f''(x)(h, h) + R_f(x, h),$$

where the remainder has an appropriate “second order” property analogous to the “first order” property of the remainder the definition of the first derivative. If the second derivative term in the last equation just given looks weird, remember that the second derivative, in this approach, is supposed to be a (symmetric) bilinear function on $\mathbb{R}^n \times \mathbb{R}^n$.

Give this appropriate definition and “remainder” property of the function $R_f$. Explain explicitly how Theorem 14.21 proves that under the assumptions of this Theorem, the second derivative, as defined here, exists and is given by the formula you obtained in part (a).

(c) The purpose of this part is to make sure you see the connection of all this abstract stuff with the second derivative as defined in elementary calculus. Suppose $f$ is a real-valued function of a real variable which has appropriately smooth second derivative (don’t worry about the continuity properties). Explain the connection between the elementary calculus second derivative, $f''(x)$, and the second derivative as defined above – i.e., show how the elementary calculus second derivative determines the bilinear function which is the second derivative as defined above.