Suppose $A$ is a subset of $\mathbb{R}^n$ and $w$ is an element of $\mathbb{R}^n$.
To prove:

1. $A$ is open iff $w + A$ is open,
and
2. $A$ is closed iff $w + A$ is closed.

**Proof.** Given the order of problems in the book, we will first prove (1) and then prove (2). It might, however, be easier to prove (2) first (using sequences) and then prove (1) (using complements).

First, some simple algebraic facts about translation (which, of course, need to be proved).

- $v$ is an element of $w + A$ iff $v - w$ is an element of $A$,
- for all subsets $C$ of $\mathbb{R}^n$, $A$ is a subset of $C$ iff $w + A$ is a subset of $w + C$,
- $-w + (w + A) = A$
- $(w + A)^c = w + A^c$
and
- for every $u$ and $r$, $B(w + u, r) = w + B(u, r)$,

Now, suppose $A$ is open. To prove $w + A$ is open, consider at point $v$ in $w + A$.

By definition of translation, we can find $a$ in $A$ such that $v = w + a$.
Since $A$ is open, there exists an open ball $B(a, r)$ centered at $a$ contained in $A$.
So $B(w + a, r) = w + B(u, r)$ is a subset of $w + A$ by the algebraic facts mentioned above.
This shows that $v$ is an interior point of $A$. Since every point of $w + A$ is an interior point of $w + A$, $w + A$ is open.

Next, suppose $w + A$ is open. It follows from what was just proved that the translation $-w + (w + A)$ of $w + A$ is open. Since $A = -w + (w + A)$, $A$ is open.

The fact that $A$ is closed iff $w + A$ is closed is a consequence of the last algebraic fact mentioned above and the fact that a set is closed iff its complement is open. For example,

- $A$ is closed
  iff
  $A^c$ is open
iff (by what was just proved for translation of open sets)
  $w + A^c$ is open
iff (since $(w + A)^c = w + A^c$)
  $(w + A)^c$ is open
iff
  $w + A$ is closed.