Math 250A (Fall 2008) - Exam III **Name:**

Instructions:

- Show all work clearly in order to get full credit. Points can be taken off if it is not clear to see how you arrived at your answer (even if the final answer is correct).
- When you do use your calculator, sketch all relevant graphs and explain all relevant mathematics.
- Circle your final answers.
- Please keep your written answers brief; be clear and to the point.
- This test has 5 problems (plus an extra credit problem) and is worth 100 points, plus extra credit at the end. It is your responsibility to make sure that you have done all the problems!

1. (*25 points*) Pick two out of the following three problems to solve (12.5 points each). Indicate which problems will count towards your grade by circling (a), (b), and/or (c) below.

(**a**) $$\frac{dg}{d\theta} = \frac{1}{14\theta - 7\theta^2}$$

$$dg = \frac{d\theta}{7(2-\theta)\theta} \Rightarrow g + C = \frac{1}{7}\int\left(\frac{1/2}{\theta} + \frac{1/2}{2-\theta}\right)d\theta$$

$$= \frac{1}{14}\left[\ln|\theta| - \ln|2-\theta|\right] = \frac{1}{14}\ln\left|\frac{\theta}{2-\theta}\right|$$

So $\boxed{g(\theta) = \frac{1}{14}\ln\left|\frac{\theta}{2-\theta}\right| + K}$ where K is an arbitrary constant

(b) $\dfrac{dh}{dw} = \dfrac{1}{\sin(6w)+1}$ with initial condition $h(3\pi/4) = 0$

$h + C = \displaystyle\int \frac{dw}{1+\sin(6w)}$ Let $t = \tan(3w)$ $dt = (1+t^2)3\,dw$

$\sin(6w) = \dfrac{2t}{1+t^2}$

$$\int \frac{dw}{1+\sin(6w)} = \int \frac{1}{3(1+t^2)} \frac{1}{1+\frac{2t}{1+t^2}} dt = \int \frac{dt}{3(1+t^2+2t)} = \frac{1}{3}\int \frac{dt}{(1+t)^2}$$

$$= -\frac{1}{3}\frac{1}{1+t} = -\frac{1}{3}\frac{1}{1+\tan(3w)}$$

So $h = -\dfrac{1}{3}\dfrac{1}{1+\tan(3w)} + C$. When $w = \dfrac{3\pi}{4}$, we have $h=0$.

Thus $0 = -\dfrac{1}{3}\dfrac{1}{1+\tan(3\frac{3\pi}{4})} + C = -\dfrac{1}{3}\dfrac{1}{1+1} + C \Rightarrow C = \dfrac{1}{6}$.

Therefore, $\boxed{h = \dfrac{1}{6} - \dfrac{1}{3}\dfrac{1}{1+\tan(3w)}}$

(c) $\displaystyle\int \frac{t^2}{\sqrt{4-t^2}}\,dt$

Let $t = 2\sin(\theta)$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Then, $dt = 2\cos(\theta)\,d\theta$ and

$$\int \frac{t^2}{\sqrt{4-t^2}}dt = \int \frac{4\sin^2(\theta)}{\sqrt{4-4\sin^2(\theta)}} 2\cos(\theta)\,d\theta = \int \frac{4\sin^2(\theta)\cdot 2\cos(\theta)}{2\sqrt{\cos^2(\theta)}}d\theta$$

Since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos(\theta) \geq 0$ and $\sqrt{\cos^2(\theta)} = \cos(\theta)$. Thus,

$$\int \frac{t^2}{\sqrt{4-t^2}}dt = \int 4\sin^2(\theta)\,d\theta = 4\int \sin^2(\theta)\,d\theta.$$

Now we look for $\displaystyle\int \sin^2(\theta)\,d\theta = -\sin(\theta)\cos(\theta) - \int -\cos(\theta)\cos(\theta)\,d\theta$

$$= -\sin(\theta)\cos(\theta) + \int \cos^2(\theta)\,d\theta$$

$$= -\sin(\theta)\cos(\theta) + \int [1-\sin^2(\theta)]\,d\theta$$

$$\Rightarrow 2\int \sin^2(\theta)\,d\theta = -\sin(\theta)\cos(\theta) + \int d\theta = \theta - \sin(\theta)\cos(\theta) + C$$

So $\displaystyle\int \frac{t^2}{\sqrt{4-t^2}}dt = 2\theta - 2\sin(\theta)\cos(\theta) + K = 2\arcsin\left(\frac{t}{2}\right) - t\sqrt{1-\left(\frac{t}{2}\right)^2} + C$

i.e. $\boxed{\displaystyle\int \frac{t^2}{\sqrt{4-t^2}}dt = 2\arcsin\left(\frac{t}{2}\right) - \frac{t}{2}\sqrt{4-t^2} + C}$

2. (*20 points*) Consider the differential equation

$$\frac{dy}{d\nu} = (y-5)^{3/4}$$

(a) Suppose we have the initial condition $y(2) = 5$. Does a solution to the differential equation exist? If so, is it unique? Explain.

Let $g(y) = (y-5)^{3/4}$.

Since $g(y)$ is continuous near $y=5$ (the initial condition), a solution exists. Because $(y-5)^{3/4}$ is not differentiable at $y=5$, we cannot conclude for uniqueness.

In fact, $y=5$ is a solution, and we can find another one by separating variables:

$$\frac{dy}{(y-5)^{3/4}} = d\nu \Rightarrow 4\,(y-5)^{1/4} = \nu + C \quad \text{so} \quad y = 5 + \left(\frac{\nu+c}{4}\right)^4.$$

If we set $c=-2$, we have $y(2)=5$. There are, therefore 2 solutions going through $(2,5)$: $y=5$ and $y = 5 + \left(\frac{\nu-2}{4}\right)^4$.

(b) Can you give an initial condition for which a solution both exists and is unique? Explain how this is different from the previous part.

In fact, g is continuously differentiable everywhere except at $y=5$. So as long as the initial condition is not of the form $y(x_0)=5$ and as long as $(y-5)^{3/4}$ is well defined, the theorem for existence & uniqueness will apply.

We can for instance choose $y(0)=8$.

g is continuous near $y=8$ and $\frac{dg}{dy} = \frac{3}{4}(y-5)^{-1/4}$ is continuous as well near $y=8$. There will therefore exist a unique solution to $y'=g(y)$ passing through $(0,8)$.

3. (*25 points*) The goal of this problem is to use Euler's method to solve the following differential equation

$$\frac{dy}{dx} = x\,e^{-y}.$$

(a) Write the formula relating y_{i+1} to y_i, x_i, and Δx, in Euler's method. Explain where this formula comes from.

$$y_{i+1} = y_i + \Delta x \; g(x_i, y_i) = y_i + \Delta x \; x_i \, e^{-y_i}$$

This is obtained by writing

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} x\, e^{-y(x)}\, dx$$

and approximating the integral by $x_i\, e^{-y_i(x_i)} \Delta x$, i.e. its left-rule approximation.

(b) Consider the interval $x = [-1, 0]$ with initial condition $y(-1) = 0$. Use Euler's method to estimate $y(0)$ using a step-size of $\Delta x = 0.25$. Write your answers in the table below.

Euler's method estimates for $dy/dx = xe^{-y}$ where $y(-1) = 0$			
step i	x_i	y_i	y_i (exact; see part (c))
0	-1	0	0
1	-0.75	-0.25	-0.24686
2	-0.5	-0.49	-0.47000
3	-0.25	-0.69	-0.63252
4	0	-0.82	-0.69315
5			

Briefly explain how you found each of the approximated y_i's:

$x_0 = -1 \qquad y_0 = 0$

$x_1 = -1 + \Delta x = -0.75 \qquad y_1 = y_0 + \Delta x \; x_0 \, e^{-y_0} = 0 + 0.25(-1)e^{-0} = -0.25$

$x_2 = -0.5 \qquad y_2 = y_1 + \Delta x \; x_1 \, e^{-y_1} = -0.25 + 0.25(-0.75)\, e^{0.25} \simeq -0.49075$

$x_3 = -0.25 \qquad y_3 = y_2 + \Delta x \; x_2 \, e^{-y_2} \simeq -0.49075 + 0.25(-0.5)\, e^{0.49075}$

$\simeq -0.69494$

$x_4 = 0 \qquad y_4 = y_3 + \Delta x \; x_3 \, e^{-y_3} \simeq -0.69494 + 0.25(-0.25)\, e^{0.69494}$

$\simeq -0.82016$

(c) An exact solution to this differential equation is $y(x) = \ln\left(\frac{x^2+1}{2}\right)$. Is your value for $y(0)$ and over- or under-estimate? Explain.

The value for $y(0)$ is an underestimate since it is less than $-\ln(2) \simeq -0.69315$.

Note that this can be expected since
$$y'' = e^{-y} - y' x e^{-y} = e^{-y}(1 - x^2 e^{-y})$$

With $y(x) = \ln\left(\frac{x^2+1}{2}\right)$, $e^{-y} = e^{\ln\left(\frac{2}{x^2+1}\right)} = \frac{2}{x^2+1}$, and $1 - x^2 e^{-y} = 1 - \frac{2x^2}{x^2+1} = \frac{1-x^2}{1+x^2} \geqslant 0$ for $-1 \leqslant x \leqslant 0$, i.e. the exact solution is concave up. The first step in Euler's method is therefore an underestimate, and one can expect $y(0)$ to be underestimated as well.

4. (*10 points*) Remember that the definition of the *hyperbolic sine* and *cosine* is $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x + e^{-x}}{2}$ respectively. Show that

$$\sinh(a+b) = \sinh(a)\cosh(b) + \sinh(b)\cosh(a)$$

$\sinh(a)\cosh(b) + \sinh(b)\cosh(a)$

$$= \frac{e^a - e^{-a}}{2} \cdot \frac{e^b + e^{-b}}{2} + \frac{e^b - e^{-b}}{2} \cdot \frac{e^a + e^{-a}}{2}$$

$$= \frac{1}{4}\left[\underline{e^{a+b}} - \cancel{e^{-a+b}} + \cancel{e^{a-b}} - \underline{\underline{e^{-a-b}}} + \underline{e^{b+a}} - \cancel{e^{-b+a}} + \cancel{e^{b-a}} - \underline{\underline{e^{-a-b}}}\right]$$

$$= \frac{1}{4}\left[2e^{a+b} - 2e^{-a-b}\right] = \frac{1}{2}\left[e^{a+b} - e^{-(a+b)}\right]$$

$$= \sinh(a+b)$$

5. (*20 points*) Consider the differential equation

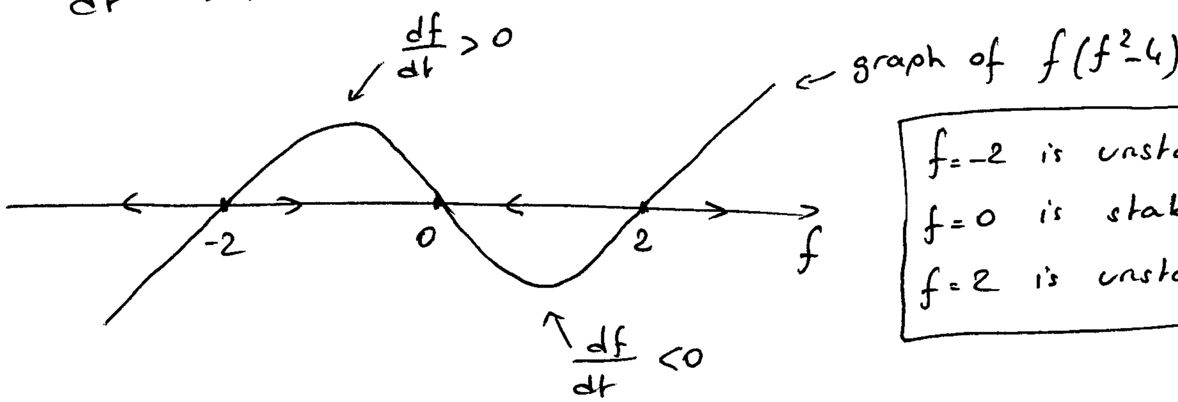
$$\frac{df}{dt} = bf(f^2 - 4)$$

(a) Find all the equilibrium points. Explain how you found them.

Equilibria are such that $\frac{df}{dt} = 0$, i.e. $b\,f\,(f^2-4) = 0$,
i.e. $f = 0$ or $f = \pm 2$ (provided $b \neq 0$).

(b) Assume $b = 1$ and draw a phase-line for this system. Use this to classify the stability of each of the equilibrium points. Make sure to clearly explain how you used the phase-line to determine the stability.

$b = 1$ $\quad \frac{df}{dt} = f\,(f^2-4)$



$f = -2$ is unstable
$f = 0$ is stable
$f = 2$ is unstable

The arrows indicate whether solutions increase (arrows pointing to the right) or decrease (arrows pointing to the left).
If arrows point towards an equilibrium, then it is stable.
If they point away, it is unstable.

(c) Provide a brief explanation of what it means for an equilibrium point to be *stable* or *unstable*.

An equilibrium is stable if for initial conditions close enough to the equilibrium, all solution curves converge towards the equilibrium.

An equilibrium is unstable if it is not stable, i.e. if no matter how close to the equilibrium we start, we can always find initial conditions such that the corresponding solution curves will move away from the equilibrium.

Extra Credit (10 points) What happens to the equilibria as the parameter b is varied? Draw a bifurcation diagram and indicate all the important features including the stability of the equilibria.

As b is varied, the equilibria remain the same, provided $b \neq 0$.

When $b = 0$, all constant values of f correspond to equilibria.

When $b < 0$, the phase line becomes 

i.e. -2 and 2 are stable and 0 is unstable.

The bifurcation diagram is then as follows: