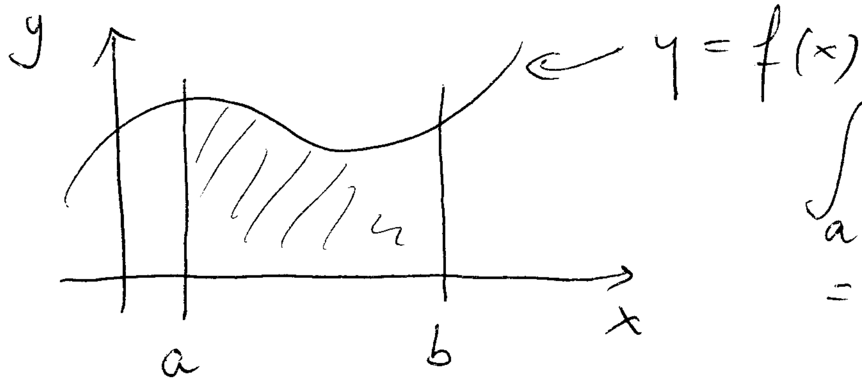
Improper Integrals



$$\int_a^b f(x)\,dx$$
= area under the graph of f between a & b

What happens when $b \to \infty$?

<u>Example</u>: $\int_1^b dx = \left[x\right]_1^b = (b-1)$

As $b \to \infty$, $b-1 \to \infty$, i.e.

$$\lim_{b \to \infty} \int_1^b dx = \infty$$

i.e. $\int_1^\infty dx$ <u>diverges</u>.

<u>Definition</u>: $\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$

$$\int_{-\infty}^c f(x)\,dx = \lim_{a \to -\infty} \int_a^c f(x)\,dx$$

Other "options"

- $$\int_{-\infty}^{c} f(x)\,dx = -\int_{c}^{-\infty} f(x)\,dx$$

$$= -\lim_{b\to-\infty} \int_{c}^{b} f(x)\,dx$$

$$= \lim_{b\to-\infty} \left(-\int_{c}^{b} f(x)\,dx\right)$$

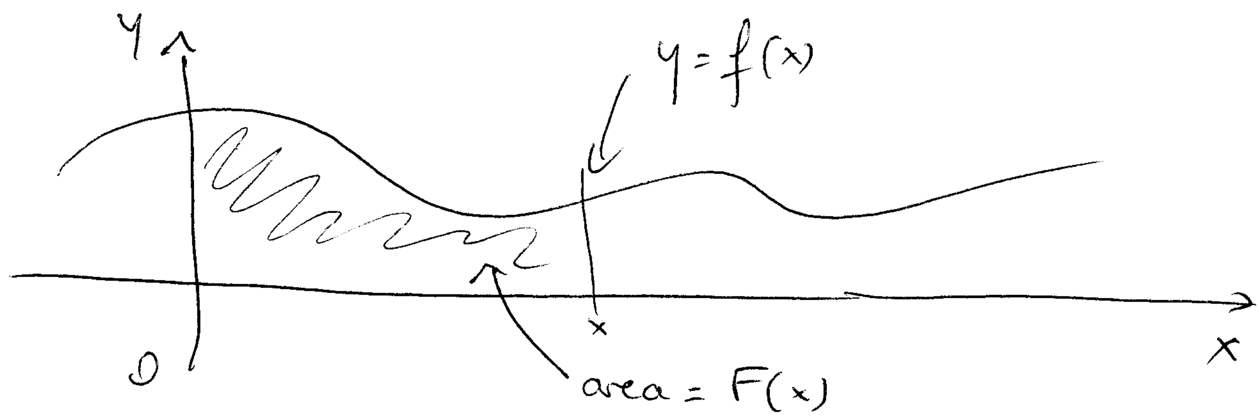
$$= \lim_{b\to-\infty} \int_{b}^{c} f(x)\,dx$$

- $$\int_{-\infty}^{c} f(x)\,dx = \int_{+\infty}^{-c} -f(-y)\,dy \qquad y=-x$$

$$= \int_{-c}^{+\infty} f(-y)\,dy = \lim_{b\to+\infty} \int_{-c}^{b} f(-y)\,dy \qquad x=-$$

$$= \lim_{b\to+\infty} \int_{c}^{-b} -f(x)\,dx = \lim_{b\to+\infty} \int_{-b}^{c} f(x)\,dx$$

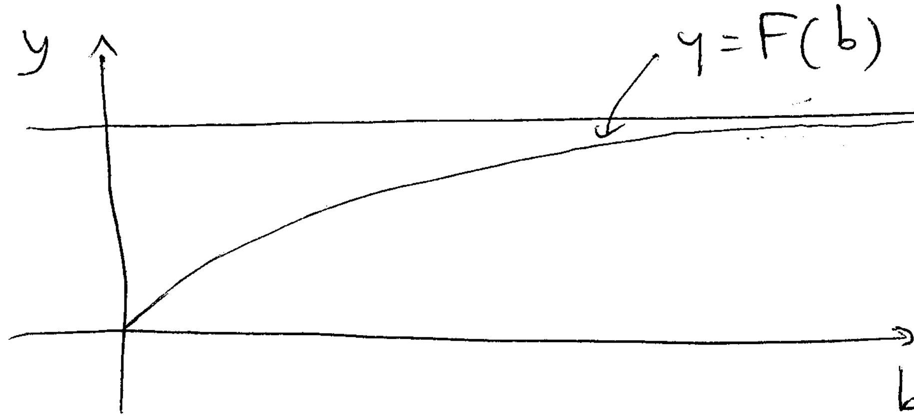
$$= \lim_{a\to-\infty} \int_{a}^{c} f(x)\,dx$$

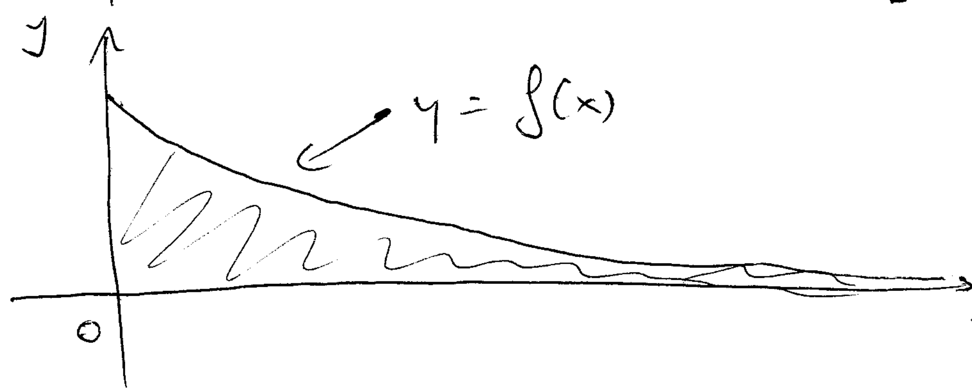


$$\int_0^\infty f(x)\,dx$$

Let $F(x) = \int_0^x f(s)\,ds$

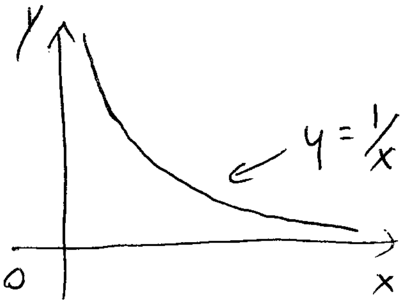
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx = \lim_{b\to\infty} F(b)$$





Remark: if $f(x) \to 0$ as $x \to \infty$, it does not necessarily mean that $\int_1^\infty f(x)\,dx$ converges.

Example: $\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x}$



$= \lim_{b\to\infty} \Big[\ln|x| \Big]_1^b$

$= \lim_{b\to\infty} \big[\ln(b) - 0 \big]$

$= \lim_{b\to\infty} \ln(b)$ diverges