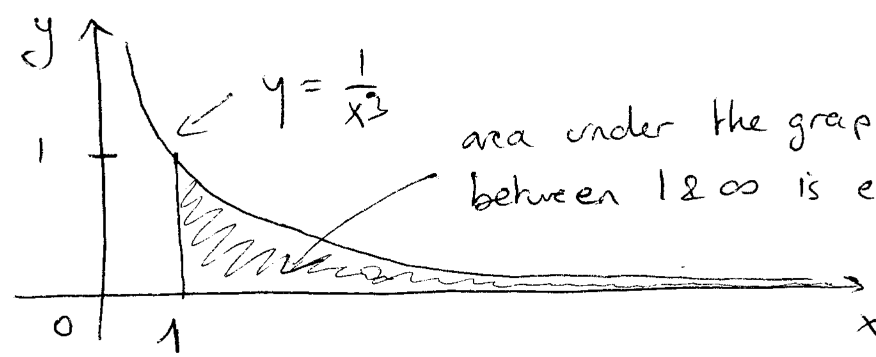
Improper integrals (continued)

Example 1
$$\int_1^\infty e^x\,dx = \lim_{b\to\infty}\int_1^b e^x\,dx$$
$$= \lim_{b\to\infty}\left[e^x\right]_1^b = \lim_{b\to\infty}\left(e^b - e\right) = +\infty$$

So $\int_1^\infty e^x\,dx$ diverges.

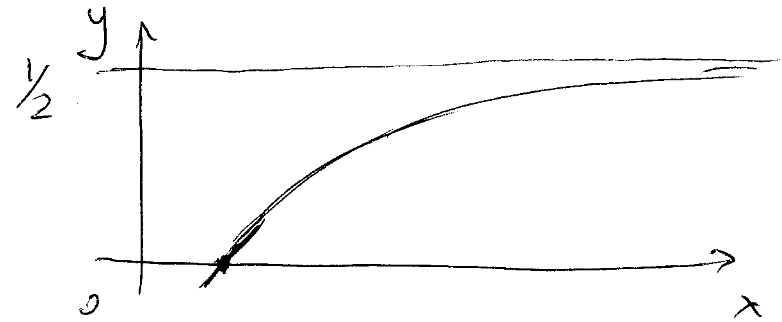
Example 2:
$$\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^3} = \lim_{b\to\infty}\left[\frac{-1/2}{x^2}\right]_1^b$$
$$= \lim_{b\to\infty}\left[-\frac{1}{2}\left(\frac{1}{b^2}-1\right)\right] = \frac{1}{2}$$

So $\int_1^\infty \frac{dx}{x^3}$ converges and is equal to $\frac{1}{2}$.



area under the graph of $\frac{1}{x^3}$ between 1 & ∞ is equal to $\frac{1}{2}$.

$$\int_2^\infty \frac{dx}{x^3} = \underbrace{\int_2^1 \frac{dx}{x^3}}_{\text{finite}} + \underbrace{\int_1^\infty \frac{dx}{x^3}}_{\text{converges}} \qquad \text{converges}$$



$$F(x) = \int_1^x \frac{1}{s^3}\, ds$$

$$F'(x) = \frac{1}{x^3}$$

$$F''(x) = \frac{-3}{x^4}$$

$$F(x) = -\frac{1}{2}\left(\frac{1}{x^2} - 1\right) \quad \text{as} \quad x \to \infty, \quad F(x) \to \frac{1}{2}$$

Example 3 :

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} = \lim_{b\to\infty} \left[\frac{x^{-p+1}}{1-p}\right]_1^b \quad \text{if } p \neq 1$$

$$= \lim_{b\to\infty} \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)$$

If $p > 1$ then $\dfrac{1}{b^{p-1}} \to 0$ as $b \to \infty$

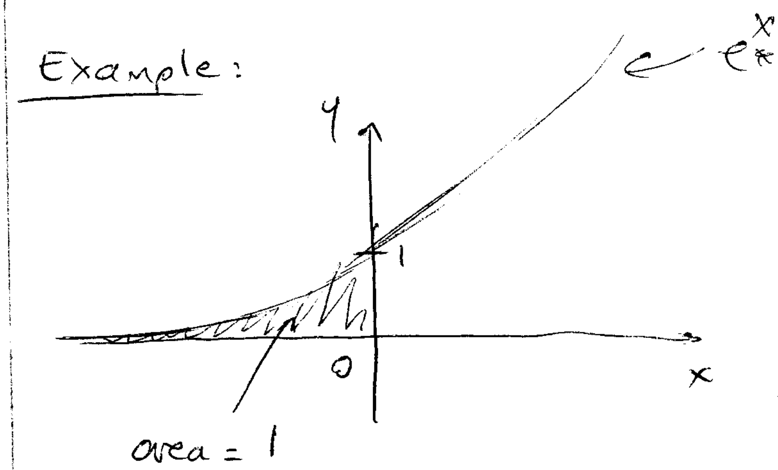
and $\displaystyle\int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1}$

If $p < 1$ then $\dfrac{1}{b^{p-1}} = b^{1-p} \to \infty$ as $b \to \infty$

and $\displaystyle\int_1^\infty \frac{dx}{x^p}$ diverges.

If $p = 1$ $\displaystyle\int_1^b \frac{dx}{x} = \left[\ln|x|\right]_1^b = \ln|b| \to \infty$

Example:

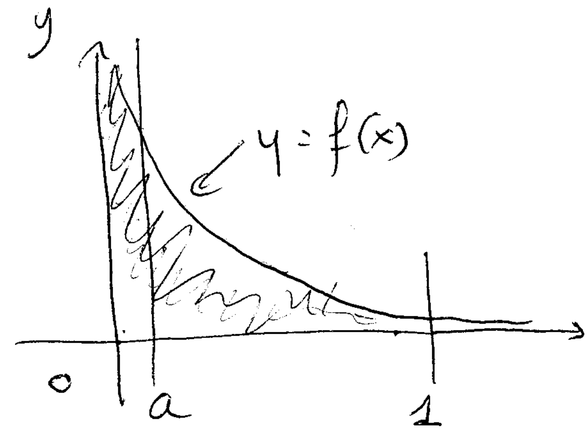


$$\int_{-\infty}^{0} e^x\,dx = \lim_{b\to-\infty} \int_b^0 e^x\,dx$$

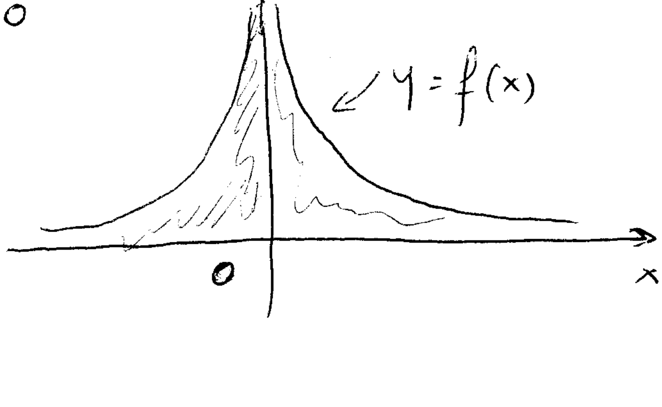
$$= \lim_{b\to-\infty} \Big[e^x\Big]_b^0 = \lim_{b\to-\infty} (1 - e^b)$$

$$= 1$$

2. Integrals for which the integrand diverges at some point in the interval of integration (or on the boundary).



$$\int_0^1 f(x)\,dx = \lim_{a\to 0^+} \int_a^1 f(x)\,dx$$



$$\int_{-1}^{1} f(x)\,dx$$

$$= \underbrace{\int_{-1}^{0} f(x)\,dx}_{\text{converges?}} + \underbrace{\int_0^1 f(x)\,dx}_{\text{converges?}}$$

<u>Example 1</u>:

$$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{b \to 2} \int_0^b \frac{dx}{\sqrt{4-x^2}}$$