Improper integrals (continued)

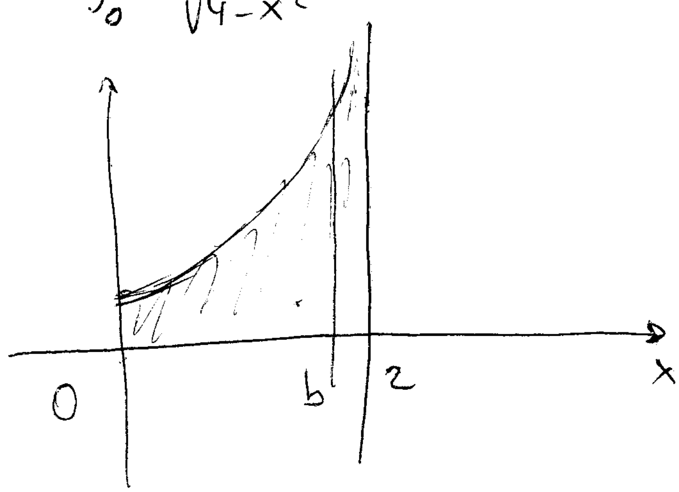
Example : $$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \int_0^2 \frac{dx}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

$$= \lim_{b\to 2^-} \int_0^b \frac{dx}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}$$

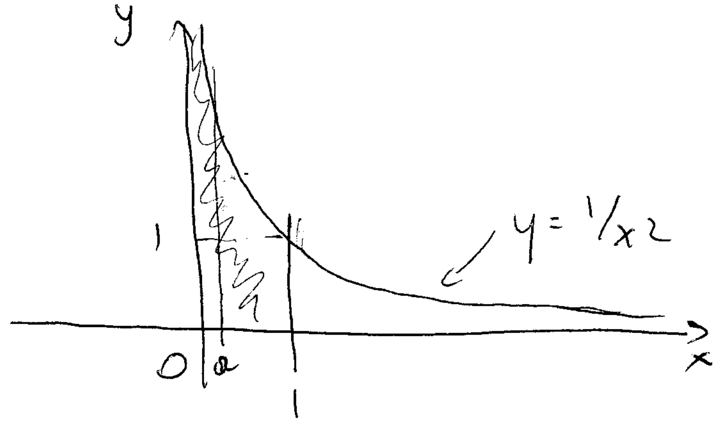
$$= \lim_{b\to 2^-} \left[\arcsin\left(\frac{x}{2}\right)\right]_0^b = \lim_{b\to 2^-} \left(\arcsin\left(\frac{b}{2}\right) - 0\right)$$

$$= \lim_{b\to 2^-} \arcsin\left(\frac{b}{2}\right) = \frac{\pi}{2}$$

So $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$ converges.



Example 2 : $$\underbrace{\int_{-1}^1 \frac{dx}{x^2}}_{\text{diverges}} \Longleftarrow \; = \int_{-1}^0 \frac{dx}{x^2} + \underbrace{\int_0^1 \frac{dx}{x^2}}_{\text{diverges}}$$



$$\int_0^1 \frac{dx}{x^2} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{x^2} = \lim_{a \to 0^+} \left[\frac{-1}{x} \right]_a^1$$

$$= \lim_{a \to 0^+} \left(\frac{1}{a} - 1 \right) = +\infty \quad \text{diverges}$$

Typical mistake: $\displaystyle\int_{-1}^1 \frac{dx}{x^2} = \left[\frac{-1}{x} \right]_{-1}^1 = -1 - (1) = -2$

Example 3: $\displaystyle\int_0^1 \frac{dx}{x^p}$

If $p < 0$, e.g. $p = -1$ $\displaystyle\int_0^1 \frac{dx}{x^{-1}} = \int_0^1 x \, dx$

converges.

By symmetry, we expect $\displaystyle\int_0^1 \frac{dx}{x^p}$ to diverge for $p \geq 1$ and to converge for $p < 1$.

Calculation: $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x^p} = \lim_{a\to 0^+} \int_a^1 x^{-p}\,dx$

$$= \lim_{a\to 0^+} \left[\frac{x^{-p+1}}{1-p}\right]_a^1 \qquad p \neq 1$$

$$= \frac{1}{1-p} \lim_{a\to 0^+} \left[\frac{1}{x^{p-1}}\right]_a^1 = \frac{1}{1-p} \lim_{a\to 0^+} \left[\frac{-1}{a^{p-1}} + 1\right]$$

- diverges if $p>1$
- converges to $\frac{1}{1-p}$ if $p<1$

If $p=1$ $\displaystyle\int_a^1 \frac{dx}{x} = \left[\ln|x|\right]_a^1 = -\ln|a| \to +\infty$ as $a \to 0^+$