Improper integrals (continued)

<u>Example 4:</u>

$$\int_{-1}^{1} \frac{du}{u} = \int_{-1}^{0} \frac{du}{u} + \underbrace{\int_{0}^{1} \frac{du}{u}}_{\text{diverges because } p = 1 \geq 1}$$

<u>Example 5:</u>

$$\int_{0}^{2} \frac{dx}{\sqrt{4-x^2}}$$

Let $u = 4 - x^2 \qquad du = -2x\,dx$

$x^2 = 4 - u$

$x = \sqrt{4-u}$

$$\int_{0}^{2} \frac{dx}{\sqrt{4-x^2}} = \int_{4}^{0} \frac{1}{\sqrt{u}} \; \frac{-du}{2\sqrt{4-u}} = \int_{0}^{4} \frac{du}{2\sqrt{u}\sqrt{4-u}}$$

3. More on the idea of convergence

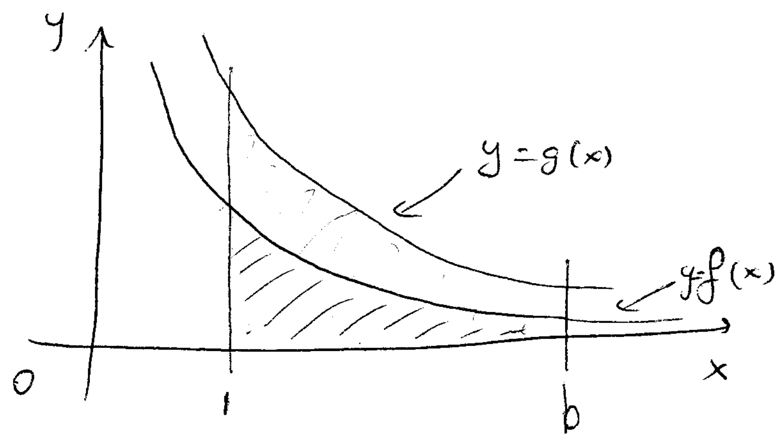
$$\int_{0}^{\infty} x e^{-x}\,dx = \lim_{b\to\infty} \int_{0}^{b} x e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[\left[-x e^{-x}\right]_{0}^{b} - \int_{0}^{b} -e^{-x}\,dx \right] = \lim_{b\to\infty} \left[-xe^{-x} - e^{-x}\right]_{0}^{b}$$

$$= \lim_{b\to\infty} \left(-be^{-b} - e^{-b} + 1\right) = 1$$

$$F(x) = \int_0^x s e^{-s}\, ds \qquad F'(x) = x e^{-x} > 0 \quad \text{for } x > 0$$

4. Comparison of improper integrals



$$0 \leq f(x) \leq g(x)$$

$$\int_1^\infty g(x)\, dx = \lim_{b \to \infty} \int_1^b g(x)\, dx$$

$$\int_1^b f(x)\, dx \leq \int_1^b g(x)\, dx$$

As $b \to \infty$, $\displaystyle\int_1^\infty f(x)\, dx \leq \int_1^\infty g(x)\, dx$

Example 1: $\displaystyle\int_1^\infty \frac{dx}{1+x^3}$

Step 1: As $x \to \infty$ $\dfrac{1}{1+x^3} \sim \dfrac{1}{x^3}$ so we expect convergence

Step 2: Use a comparison:

for $x > 0$, $x^3 \leq 1 + x^3 \Rightarrow \dfrac{1}{1+x^3} \leq \dfrac{1}{x^3}$.

So $\displaystyle\int_1^\infty \frac{dx}{1+x^3} \leq \int_1^\infty \frac{dx}{x^3}$ converges since $3=p>1$

Example 2 : $\displaystyle\int_4^\infty \frac{3+\sin(\alpha)}{\alpha}\, d\alpha$

Step 1 : $\dfrac{3+\sin(\alpha)}{\alpha}$ "goes like" $\dfrac{\text{Constant}}{\alpha}$ so we expect the integral to diverge since $p=1$.

Step 2 :

$$2 \leq 3+\sin(\alpha) \leq 4 \overset{\alpha>0}{\Rightarrow} \underbrace{\frac{2}{\alpha} \leq \frac{3+\sin(\alpha)}{\alpha}} \leq \frac{4}{\alpha}$$

So $\displaystyle\underbrace{\int_4^\infty \frac{2}{\alpha}\, d\alpha}_{\text{diverge}} \leq \int_4^\infty \frac{3+\sin(\alpha)}{\alpha}\, d\alpha$

so $\displaystyle\int_4^\infty \frac{3+\sin(\alpha)}{\alpha}\, d\alpha$