Sequences & series (continued)

4. Convergence of series (continued)

Example 1: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

$$a_n = \frac{(-1)^{n-1}}{n} \qquad \frac{a_{n+1}}{a_n} = \overbrace{\frac{(-1)^{n-1+1}}{n+1}}^{a_{n+1}} \cdot \overbrace{\frac{n}{(-1)^{n-1}}}^{1/a_n}$$

$$= -\frac{n}{n+1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1}\right| = 1$$

$$= \lim_{n\to\infty} \left|\frac{1}{1+\frac{1}{n}}\right|$$

So the ratio test does not allow us to conclude.

Can we use the alternating series test instead?

$b_n = \frac{1}{n}$ The series is of the form $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0 \quad ✓$$

$$0 < b_{n+1} = \frac{1}{n+1} < b_n = \frac{1}{n} \quad ✓$$

So the series converges.

Note the series is not absolutely convergent since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad \leftarrow \text{harmonic series}$$

we know this diverges because of the integral test.

Say we want to evaluate (i.e. approximate)

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$ correctly to 3 decimal places. How many terms should we include?

We want $|S - S_n| < 5 . 10^{-4}$

It is enough to have $a_{n+1} < 5 . 10^{-4}$ since we know that

$$|S - S_n| < a_{n+1}$$

i.e. $\frac{1}{n+1} < 5 . 10^{-4} \Rightarrow n+1 > \frac{10^4}{5} = 2\,10^3$

<u>Example 2</u>: $$\sum_{n=0}^{\infty} \frac{2}{\sqrt{2+n}} = \sum_{n=1}^{\infty} \frac{2}{\sqrt{2+n}} + \sqrt{2}$$

$\int_1^{\infty} \frac{2}{\sqrt{2+x}}\, dx$ diverges since $p = 1/2$

<u>Could we have used the ratio test?</u> $a_n = \frac{2}{\sqrt{2+n}}$

$$\frac{a_{n+1}}{a_n} = \frac{2}{\sqrt{2+n+1}} \cdot \frac{\sqrt{2+n}}{2} = \sqrt{\frac{2+n}{3+n}} \longrightarrow 1$$

<u>Could we have used the limit test?</u>

Let $b_n = \frac{1}{\sqrt{n}} > 0$. We know $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

(because of the integral test)

Moreover $\frac{a_n}{b_n} = \frac{2}{\sqrt{2+n}} \cdot \sqrt{n} = 2\sqrt{\frac{n}{2+n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 2$$

Since $\sum_{n=1}^{\infty} b_n$ diverges, so does $\sum_{n=1}^{\infty} a_n$,

<u>Example 3</u>: $\sum_{n=1}^{\infty} \frac{n+2^n}{n\,2^n}$

$$= \sum_{n=1}^{\infty} \left(\frac{n}{n\,2^n} + \frac{2^n}{n\,2^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{2^n} + \frac{1}{n}\right) \geq \sum_{n=1}^{\infty} \frac{1}{n}$$

Then use the integral test to show that

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

So $\sum_{n=1}^{\infty} \frac{n+2^n}{n\,2^n}$ diverges.