Sequences & series (continued)

4. Tests for convergence (continued)

Example 4: $\displaystyle\sum_{n=5}^{\infty} \frac{n\,2^n}{3^n}$

Let $a_n = n\,\dfrac{2^n}{3^n}$

1. Does a_n go to 0 as $n \to \infty$?

$$a_n = n\left(\frac{2}{3}\right)^n$$

$$= n\,e^{n \ln(2/3)} \longrightarrow 0 \quad \text{as } n \to \infty$$

the exponential "wins"

2. Ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)\cdot(2/3)^{n+1}}{n\cdot(2/3)^n}$$

$$= \frac{n+1}{n}\cdot\frac{2}{3}$$

As $n \to \infty$ $\quad \dfrac{a_{n+1}}{a_n} \to \dfrac{2}{3} < 1$

So the series converges.

3. Could we have used the integral test?

$$\int_1^{\infty} x\left(\frac{2}{3}\right)^x dx = \int_1^{\infty} x\,e^{x\ln(2/3)}\,dx \quad \text{converges}$$

Example 5: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^n}{n^2}$

$$\frac{2^n}{n^2} = \frac{e^{n \ln(2)}}{n^2} \to \infty \quad \text{as} \quad n \to \infty$$

Since $\lim_{n \to \infty} \frac{(-1)^{n-1} 2^n}{n^2} \neq 0$, the series diverges.

Could you have used the ratio test?

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^n}{(-1)^{n-1}} \frac{2^{n+1}}{2^n} \frac{n^2}{(n+1)^2}$$

$$= -1 \cdot 2 \cdot \left(\frac{n}{n+1}\right)^2$$

$$\left|\frac{a_{n+1}}{a_n}\right| = 2 \left|\frac{n}{n+1}\right|^2 \longrightarrow 2 \quad \text{as} \quad n \to \infty$$

Since $2 > 1$, the series diverges.