Series expansion & linear combinations

1. Taylor series

Example 1: $f(x) = \arctan(x)$ near $x = 0$

$$f(x) = \arctan(x) \qquad f(0) = 0$$

$$f'(x) = \frac{1}{1+x^2} \qquad f'(0) = 1$$

$$f''(x) = \frac{-2x}{(1+x^2)^2} \qquad f''(0) = 0$$

$$f'''(x) = \frac{-2}{(1+x^2)^2} + \frac{(-2x)\left(-2(1+x^2)(2x)\right)}{(1+x^2)^3} \qquad f'''(0) = -2$$

Near $x = 0$

$$\arctan(x) = 0 + 1 . x + 0\,\frac{x^2}{2!} - 2\,\frac{x^3}{3!} + 0\,\frac{x^4}{4!} + O(x^5)$$

$$= x - \frac{x^3}{3} + O(x^5)$$

For now, consider that $\arctan(x)$ is such that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Find the interval of convergence.

$$a_n = (-1)^n \frac{x^{2n+1}}{2n+1} \qquad n \geq 0$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2(n+1)+1}}{x^{2n+1}} \quad \frac{2n+1}{2(n+1)+1}\right|$$

$$= \left|x^2 \frac{2n+1}{2n+3}\right| \xrightarrow[n\to\infty]{} x^2$$

So the series converges for $x^2 < 1$ i.e. its radius of convergence is 1.

Example 2: Binomial series $(1+x)^p$

$$f(x) = (1+x)^p \qquad f(0) = 1$$

$$f'(x) = p(1+x)^{p-1} \qquad f'(0) = p$$

$$f''(x) = p(p-1)(1+x)^{p-2} \qquad f''(0) = p(p-1)$$

$$\vdots$$

$$f^{(n)}(x) = p(p-1)(p-2)\cdots(p-n+1)(1+x)^{p-n} \qquad f^{(n)}(0) = \frac{p!}{(p-n)!}$$

$$= \frac{p!}{(p-n)!}(1+x)^{p-n}$$

Taylor series: $(1+x)^p = \sum_{n=0}^{\infty} \frac{p!}{(p-n)!} x^n \frac{1}{n!}$

$$a_n = \frac{p!}{(p-n)!\,n!} x^n$$

Radius of convergence:

Note that if p is a positive integer, we have a polynomial, not a series.

For $p \notin \mathbb{N}$, calculate the radius of convergence of the series:

$$a_n = \frac{p!}{(p-n)!\,n!} x^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{p!}{(p-n-1)!\,(n+1)!} x^{n+1} \frac{(p-n)!\,n!}{p!\,x^n}\right|$$

$$= \left|\frac{(p-n)!}{(p-n-1)!} \frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!}\right|$$

$$= \left|(p-n)\, x \frac{1}{n+1}\right| = \left|\frac{p-n}{n+1}\right| |x| \xrightarrow[n\to\infty]{} |x|$$

Radius of convergence: $\underline{1}$.