Series expansions & linear combinations (continued)

Taylor series by substitution

Find the Taylor series of $\cos(x^2)$

1. Start with the Taylor series of $\cos(x)$
2. Substitute x^2 for x to get the Taylor series for $\cos(x^2)$.

$$\cos(x) = \cos(0) - \sin(0)\frac{x}{1!} + \cos''(0)\frac{x^2}{2!} + \cos'''(0)\frac{x^3}{3!} + \cdots + \cos^{(n)}(0)\frac{x^n}{n!} + \cdots$$

Since $\sin(0) = 0$, every other term is 0.

So $$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Radius of convergence of $\cos(x)$:

$$a_n = (-1)^n \frac{x^{2n}}{(2n)!} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{2n+2}}{x^{2n}} \frac{(2n)!}{(2n+2)!}\right|$$

$$= x^2 \frac{1}{(2n+2)(2n+1)}$$

As $n \to \infty$, $\left|\frac{a_{n+1}}{a_n}\right| \to 0$ so $R = \infty$

Since $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

we have that

$$\cos(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}$$

<u>Find the Taylor series of $e^x \cos(x)$</u>

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

<u>Radius of convergence</u>: $a_n = \frac{x^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{x^n} \frac{n!}{(n+1)!}\right| = |x| \frac{1}{n+1}$$

As $n \to \infty$, $\left|\frac{a_{n+1}}{a_n}\right| \to 0$ i.e. $R = \infty$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$e^x \cos(x) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}\right)$$

Term in x^n will have a coefficient of the form $\quad (x^n = x^{p+2q})$

$$\sum_{\substack{p,q\geq 0\\ p+2q=n}} \frac{1}{p!}(-1)^q \frac{1}{(2q)!}$$

If you are interested in the first few terms, then

$$e^x\cos(x) = \left(1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\cdots\right)\left(1-\frac{x^2}{2}+\frac{x^4}{4!}+\cdots\right)$$

$$= 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots$$

$$-\frac{x^2}{2}-\frac{x^3}{2}-\frac{x^4}{4}-\cdots$$

$$+\frac{x^4}{4!}+\cdots$$

$$= 1+x-\frac{x^3}{3}+\underbrace{O(x^4)}_{\text{order of } x^4}$$

<u>Taylor series by differentiation & integration</u>

<u>Example</u>: Find the Taylor series of $\arctan(x)$.

$$\frac{d}{dx}\left(\arctan(x)\right) = \frac{1}{1+x^2}$$

We recognize the sum of a geometric

series with $a = 1$ & "x" $= -x^2$.

So $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \dots$

By integration,

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Expansion of functions onto other functions

Given a smooth function f, can we write f as an expansion onto functions in the set $\{1, x, x^2, x^3, \dots, x^n, \dots\}$.

Yes, with the Taylor series

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

assuming that $R \neq 0$.

$$f(x) = f(0) + f'(0)\, x + f''(0) \frac{x^2}{2!} + \dots + \frac{f^{(n)}(0)}{n!} x^n$$

$$= C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n + \dots$$

where $C_n = \dfrac{f^{(n)}(0)}{n!}$