Series expansions & Linear Combinations (continued)

2. Linear independence (continued)

Example 2: Are $\sin(x)$ & $\cos(x)$ linearly independent

$$c_1 \sin(x) + c_2 \cos(x) = 0 \quad \text{for all } x\text{'s}$$

Set $x = 0$ then $c_1 \cdot 0 + c_2 \cdot 1 = 0 \Rightarrow c_2 = 0$

Set $x = \frac{\pi}{2}$ then $c_1 \cdot 1 + c_2 \cdot 0 = 0 \Rightarrow c_1 = 0$

Example 3: Are $\sin(x)$, $\cos(x)$, $\sin(2x)$, $\cos(2x)$ linearly independent?

$$c_1 \sin(x) + c_2 \cos(x) + c_3 \sin(2x) + c_4 \cos(2x) = 0$$

Recall that $\int_0^{2\pi} \sin(mx) \cos(nx)\, dx = 0$

$$\int_0^{2\pi} \sin(mx) \sin(nx)\, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

$$\int_0^{2\pi} \cos(mx) \cos(nx)\, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

Here m & n are positive integers.

Start with

$$c_1 \sin(x) + c_2 \cos(x) + c_3 \sin(2x) + c_4 \cos(2x) = 0$$

Multiply by sin(x)

$$c_1 \sin^2(x) + c_2 \sin(x)\cos(x) + c_3 \sin(x)\sin(2x) + c_4 \sin(x)\cos(2x) = 0$$

Integrate between 0 & 2π

$$c_1 \int_0^{2\pi} \sin^2(x)\,dx + c_2 \int_0^{2\pi} \sin(x)\cos(x)\,dx + c_3 \int_0^{2\pi} \sin(x)\sin(2x)\,dx + c_4 \int_0^{2\pi} \sin(x)\cos(2x)\,dx = 0$$

i.e. $c_1 \pi + c_2 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 = 0$

i.e. $c_1 \pi = 0 \Rightarrow c_1 = 0$.

Similarly,

- multiplying by $\cos(x)$ & integrating between 0 & 2π gives $c_2 = 0$
- multiplying by $\sin(2x)$ " " " gives $c_3 = 0$
- multiplying by $\cos(2x)$ " " " gives $c_4 = 0$.

Therefore,

$$c_1 \sin(x) + c_2 \cos(x) + c_3 \sin(2x) + c_4 \cos(2x) = 0$$
$$\Rightarrow c_1 = c_2 = c_3 = c_4 = 0.$$

Thus, $\{\sin(x), \cos(x), \sin(2x), \cos(2x)\}$ is linearly <u>independent</u>.

For those of you taking vector calculus: how do we show that $\bar{i}, \bar{j}, \bar{k}$ are linearly indepen-dent?

Start with $c_1 \bar{i} + c_2 \bar{j} + c_3 \bar{k} = \vec{0}$

- Take the dot product with $\bar{i}$

  $\bar{i} \cdot (c_1 \bar{i} + c_2 \bar{j} + c_3 \bar{k}) = \bar{i} \cdot \vec{0} = 0$

  i.e. $c_1 \, \bar{i} \cdot \bar{i} + c_2 \, \bar{i} \cdot \bar{j} + c_3 \, \bar{i} \cdot \bar{k} = 0$

  i.e. $c_1 \cdot 1 + c_2 \cdot 0 + c_3 \cdot 0 = 0$

  i.e. $c_1 = 0$

  ($\bar{i} \cdot \bar{j} = 0$ and $\bar{i} \cdot \bar{k} = 0$ because the vectors $\bar{i}, \bar{j}, \& \bar{k}$ are orthogonal to one another).

- Similarly, taking the dot product with $\bar{j}$ gives $c_2 = 0$.
- Taking the dot product with $\bar{k}$ gives $c_3 = 0$.

<u>Example 4</u>: Are e^x & e^{-x} linearly independent?

Start with $c_1 e^x + c_2 e^{-x} = 0$ for all x's

- If e^x & e^{-x} are not linearly independent, then there exist a c_1 & a c_2 not both equal to 0 such that $c_1 e^x + c_2 e^{-x} = 0$

  if $c_1 \neq 0$ then $e^x = -\dfrac{c_2}{c_1} e^{-x}$ for all x's

  this is impossible

  if $c_2 \neq 0$ then $e^{-x} = -\dfrac{c_1}{c_2} e^x$ for all x's

  this is impossible.

- Alternatively, start with $c_1 e^x + c_2 e^{-x} = 0$ for all x's.

Set $x = 0$ $\quad c_1 + c_2 = 0 \implies c_1 = -c_2$

Set $x = 1$ $\quad c_1 e + c_2 \frac{1}{e} = 0 = -c_2 e + c_2 \frac{1}{e}$

$$\implies c_2\left(-e + \frac{1}{e}\right) = 0$$

$$\implies c_2 = 0$$

Since $c_1 = -c_2$, then $c_1 = 0$ as well so $\{e^x, e^{-x}\}$ is linearly <u>independent</u>.

<u>Example 5</u>: Is $\{e^x, e^{-x}, e^{2x}\}$ linearly independe

Start with (1) $c_1 e^x + c_2 e^{-x} + c_3 e^{2x} = 0$ for all x's

Differentiate:

$$c_1 e^x - c_2 e^{-x} + 2c_3 e^{2x} = 0 \quad (2)$$

Differentiate again!

$$c_1 e^x + c_2 e^{-x} + 4 c_3 e^{2x} = 0 \quad (3)$$

(3) − (1) $\implies 3c_3 e^{2x} = 0$ for all x's

$\implies c_3 = 0$

Then (1) becomes $c_1 e^x + c_2 e^{-x} = 0$ (4)

(2) " $c_1 e^x - c_2 e^{-x} = 0$ (5)

(4) + (5) $\implies 2c_1 e^x = 0$ for all x's

$\implies c_1 = 0$

(5) − (4) $\implies -2c_2 e^{-x} = 0$ for all x's

$\implies c_2 = 0$

So $c_1 = c_2 = 0$ i.e. $\{e^x, e^{-x}, e^{2x}\}$ is linearly <u>independent</u>.

<u>Example 6</u>: Are $1, x, x^2, (1+x)^2$ linearly independent?

$$(1+x)^2 = 1 + 2x + x^2$$

i.e. $$1 + 2x + x^2 - (1+x)^2 = 0$$

$$c_1 = 1$$
$$c_2 = 2$$
$$c_3 = 1$$
$$c_4 = -1$$

i.e. we have a linear combination of the 4 functions equal to zero, with the c_i's not zero.

Therefore, $\{1, x, x^2, (1+x)^2\}$ is linearly <u>dependent</u>.