Series expansions & linear combinations (continued)

3. Taylor series and differential equations

Example 1: Find the Taylor series of

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

near $x = 0$.

We know $e^t = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots$

Then
$$e^{-t^2} = 1 - t^2 + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \cdots + \frac{(-t^2)^n}{n!} + \cdots$$
$$= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots$$

Integrate:

$$\int_0^x e^{-t^2}\, dt = x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)\, n!} + \cdots$$

Example 2: $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \underset{j = n-1}{=} \sum_{j=0}^{\infty} (j+1)\, a_{j+1}\, x^j$$

$$y' = 2x - y \qquad y(0) = 1$$

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Substitute into ODE:

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 2x - \sum_{n=0}^{\infty} a_n x^n$$