Series expansions & linear combinations (continued)

3. Taylor series & differential equations (continued)

Example: $y' = 2x - y$

$$y = \sum_{k=0}^{\infty} a_k x^k \qquad y' = \sum_{k=1}^{\infty} a_k k x^{k-1}$$

$$= \sum_{k=0}^{\infty} a_{k+1} (k+1) x^k$$

$$\sum_{k=0}^{\infty} a_{k+1} (k+1) x^k = 2x - \sum_{k=0}^{\infty} a_k x^k$$

$$\left.\begin{array}{lll} k=0 & a_1 \cdot 1 = -a_0 & \\ k=1 & a_2 \cdot 2 = 2 - a_1 & \\ k=2 & a_3 \cdot 3 = -a_2 & \\ k=3 & a_4 \cdot 4 = -a_3 & \\ & \vdots & \\ k=n-1 & a_n \cdot n = -a_{n-1} & \end{array}\right\}$$

take the product:

$$a_3 \cdot a_4 \cdot a_5 \cdot \ldots \cdot a_n \frac{n!}{2} = a_2 \cdot a_3 \cdot a_4 \cdot \ldots \cdot a_{n-1} (-1)^{n-2}$$

Assume the a_i's are not 0 & simplify:

$$a_n \frac{n!}{2} = a_2 (-1)^{n-2}$$

$$\Rightarrow a_n = \frac{2}{n!} (-1)^n a_2$$

Now find a_0, a_1, and a_2:

$$a_1 = -a_0 ; \quad a_2 = \frac{1}{2}(2 - a_1) = 1 - \frac{a_1}{2} = 1 + \frac{a_0}{2}$$

Use initial condition to find a_0:

$1 = y(0) = a_0 \quad$ so $\quad a_0 = 1$

Then $\quad a_1 = -a_0 = -1$

$$a_2 = 1 + \frac{a_0}{2} = 1 + \frac{1}{2} = \frac{3}{2}$$

and

$$a_n = \frac{2}{n!}(-1)^n a_2 = \frac{2}{n!}(-1)^n \frac{3}{2}$$

$$= 3\,\frac{(-1)^n}{n!}$$

Substitute back into expression for y:

$$y = \sum_{h=0}^{\infty} a_h x^h$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_h x^h + \cdots$$

$$= 1 - x + \frac{3}{2}x^2 + 3\frac{(-1)^3}{3!}x^3 + \cdots + 3\frac{(-1)^h}{h!}x^h + \cdots$$

$$= 1 - x + 3\sum_{h=2}^{\infty}\frac{(-1)^h}{h!}x^h$$

$$= 1 - x + 3\left(e^{-x} - (1-x)\right) = 1 - x + 3e^{-x} - 3 + 3x$$

$$\boxed{y = 3e^{-x} - 2 + 2x},$$

Check: Solve the differential equation directly

$$y' = 2x - y \quad \text{with} \quad y(0) = 1$$

This is a 1^{st} order linear equation:

$$y' + y = 2x$$

Multiply by e^x to get

$$e^x y' + e^x y = 2xe^x$$

i.e. $\frac{d}{dx}(ye^x) = 2xe^x$

i.e. $ye^x = \int 2xe^x dx + C$

$$= 2\left[xe^x - \int e^x dx\right] + C$$

$$= 2xe^x - 2e^x + C$$

$$\Rightarrow \quad y = 2x - 2 + Ce^{-x}$$

Apply initial condition:

$$1 = y(0) = 0 - 2 + C \Rightarrow C = 3$$

Therefore $\boxed{y(x) = 2x - 2 + 3e^{-x}}$ ✓