Linear differential equations (continued)

I. Introduction & overview (continued)

Example 2: $7x\,y''' + 2y' = \cos(x)$
3rd order, linear, not with constant coefficients, non-homogeneous

Existence & uniqueness: (see theorem in notes)
Example: $y^{(4)} - x^3 y'' + 3y = 0$
$y(0) = 1$; $y'(0) = 1$; $y''(0) = 0$; $y^{(3)}(0) = 0$
Does this initial value problem have a unique solution on $[-1, 1]$?

The solution exists & is unique on $\mathbb{R}$ since all of the coefficients of this homogeneous equation are continuous functions of x.

General facts

$$y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_1(x) y' + a_0(x) y = h(x)$$

Assume we know a particular solution $y_p(x)$.
Then,

$$y_p^{(n)} + a_{n-1}(x)\, y_p^{(n-1)} + \cdots + a_1(x)\, y_p' + a_0(x) y_p = h(x)$$

Take the difference:

$$y^{(n)} - y_p^{(n)} + a_{n-1}(x)\left(y^{(n-1)} - y_p^{(n-1)}\right) + \cdots + a_1(x)(y' - y_p') + a_0(x)(y - y_p) = 0$$

[redacted]

This can be re-written as

$$(y-y_p)^{(n)} + a_{n-1}(x)\,(y-y_p)^{(n-1)} + \dots + a_1(x)\,(y-y_p)' + a_0(x)\,(y-y_p) = 0$$

This happens because the equation is <u>linear</u>!

Let $u = y - y_p$. Then u satisfies the <u>homogeneous</u> equation

$$u^{(n)} + a_{n-1}(x)\,u^{(n-1)} + \dots + a_1(x)\,u' + a_0(x)\,u = 0$$

If we know the general solution $y_h(x)$ to the homogeneous equation, then we can write $u(x) = y_h(x)$, i.e. since $u(x) = y(x) - y_p(x)$ we have

$$y_h(x) = u(x) = y(x) - y_p(x)$$

$$\Rightarrow \boxed{y(x) = y_h(x) + y_p(x)}$$