Linear differential equations with constant coefficients (continued)

Why does the method of | variation of parameters
| reduction of order

work?

Recall: Consider $ay'' + by' + cy = 0$
Assume $y_1(x)$ is a solution.
Look for another solution in the form $y_2 = u(x)\, y_1(x)$
Substitute in to get:

$$v'(ay_1) + v(2ay_1' + by_1) = 0$$

Now assume that y_1 is an exponential, i.e. $y_1(x) = e^{\lambda x}$.

Then we have $y_1' = \lambda e^{\lambda x}$ and the ODE for v is

$$v' a e^{\lambda x} + v\left(2a\lambda e^{\lambda x} + b e^{\lambda x}\right) = 0$$

i.e.

$$v' a e^{\lambda x} + v(2a\lambda + b)e^{\lambda x} = 0$$

Since $e^{\lambda x} \neq 0$, then

$$av' + v(2a\lambda + b) = 0.$$

Recall that λ solves $a\lambda^2 + b\lambda + c = 0$ but in the special case where there is only 1 root, i.e. $b^2 - 4ac = 0$.

In this case, $\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b}{2a}$,

i.e. $2a\lambda + b = 0$.

The ODE for v is then $av' = 0$, i.e. (since $a \neq 0$), $v' = 0$.

Then $v =$ constant $= C_1$ and $u = C_1 x + C_2$ (since $u' = v$).

Recall that we are looking for $\underline{\underline{1}}$ $y_2(x)$, so we can pick simple values of C_1 & C_2.

- Can I pick $C_1 = 0$? No because then $u = C_2$ and $y_2(x) = C_2 y_1(x)$ i.e. is not linearly independent from y_1.

- Pick $C_2 = 0$ and $C_1 = 1$.

Then $y_2(x) = x\, y_1(x)$.

The general solution to $ay'' + by' + cy = 0$ is, in this case, given by

$$\begin{aligned} y(x) &= C_1 y_1(x) + C_2 y_2(x) \\ &= C_1 e^{\lambda x} + C_2 x e^{\lambda x} \end{aligned}$$

<u>Remark</u>: The method of reduction of order works even if a, b and c are not constant functions $\Rightarrow$ it is extremely general for linear equations!