Linear differential equations with constant coefficients (continued)

2.1. Variation of parameters (continued)

Example 1: $y'' + y' - 2y = x$

Step 1: Find general solution to homogeneous equation.

We've already solved the homogeneous equation. Two linearly independent solutions are

$$y_1(x) = e^x \quad \text{and} \quad y_2(x) = e^{-2x}$$

Step 2: Find a particular solution to the non-homogeneous equation.

Here, we want to use the method of variation of parameters.

We look for a solution in the form

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$
$$= u_1(x)\, e^x + u_2(x)\, e^{-2x}$$

together with

$$u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x) = 0$$

i.e. $u_1'(x)\, e^x + u_2'(x)\, e^{-2x} = 0$

Here we are going to work out all of the algebra, to illustrate the method:

$$y_p'(x) = \underline{u_1' e^x} + u_1 e^x + \underline{u_2' e^{-2x}} - 2u_2 e^{-2x}$$
$$= 0 + u_1 e^x - 2u_2 e^{-2x}$$

$$y_p'' = u_1' e^x + u_1 e^x - 2u_2' e^{-2x} + 4u_2 e^{-2x}$$

Plug back into $y'' + y' - 2y = x$

$$y_p'' + y_p' - 2y_p = x$$

$$\Leftrightarrow u_1' e^x + u_1 e^x - 2u_2' e^{-2x} + 4u_2 e^{-2x} + u_1 e^x - 2u_2 e^{-2x} - 2\left(u_1 e^x + u_2 e^{-2x}\right) = x$$

$$\Leftrightarrow u_1\left(e^x + e^x - 2e^x\right) + u_2(4 - 2 - 2)e^{-2x} + u_1' e^x - 2u_2' e^{-2x} = x$$

$$\Leftrightarrow u_1' e^x - 2u_2' e^{-2x} = x.$$

So we have to solve the above equation, together with $u_1' e^x + u_2' e^{-2x} = 0$.

$$\begin{cases} u_1' e^x + u_2' e^{-2x} = 0 & (1) \\ u_1' e^x - 2u_2' e^{-2x} = x & (2) \end{cases}$$

<u>Check</u>: $\delta = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix} = -2e^x e^{-2x} - e^x e^{-2x} = -3e^{-x} \neq 0$

So, as expected, there is a unique solution to this system, (u_1', u_2').

$2(1) + (2) \Rightarrow 3u_1' e^x = x$ i.e. $u_1' = \dfrac{xe^{-x}}{3}$

$(1) - (2) \Rightarrow 3u_2' e^{-2x} = -x$

i.e. $u_2' = \dfrac{-x}{3} e^{2x}$

Integrate to find u_1 & u_2

$$u_1 = \int \frac{x e^{-x}}{3}\, dx = \frac{1}{3}\left(-e^{-x} x - \int -e^{-x} dx\right)$$
$$= \frac{1}{3}\left(-xe^{-x} + \int e^{-x} dx\right) = \frac{1}{3}\left(-xe^{-x} - e^{-x} + C_1\right)$$

$$u_2 = \int \frac{-x}{3} e^{2x}\, dx = -\frac{1}{3} \int x e^{2x}\, dx$$
$$= \frac{-1}{3}\left(\frac{1}{2} e^{2x} x - \int \frac{1}{2} e^{2x} dx\right)$$
$$= -\frac{1}{3}\left(\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C_2\right)$$
$$= \frac{-1}{6} x e^{2x} + \frac{1}{12} e^{2x} - \frac{1}{3} C_2$$

Plug back into y_p:

$$y_p = u_1 e^{x} + u_2 e^{-2x}$$
$$= \left(-\frac{x}{3} e^{-x} - \frac{1}{3} e^{-x} + \frac{C_1}{3}\right) e^{x} + \left(\frac{-1}{6} x e^{2x} + \frac{1}{12} e^{2x} - \frac{1}{3} C_2\right) e^{-2x}$$
$$= -\frac{x}{3} - \frac{1}{3} + K_1 e^{x} - \frac{x}{6} + \frac{1}{12} + K_2 e^{-2x}$$
$$= -\frac{x}{2} - \frac{1}{4} + K_1 e^{x} + K_2 e^{-2x}$$

Since we only need one particular solution, choose $K_1 = K_2 = 0$ (which is the same as setting C_1 & C_2 to 0 when looking for u_1 & u_2). Then, $y_p = -\frac{x}{2} - \frac{1}{4}$ is one particular solution.

Step 3 : Write down the general solution to the non-homogeneous equation:

$$y(x) = y_h(x) + y_p(x)$$

$$= c_1 e^{x} + c_2 e^{-2x} - \frac{x}{2} - \frac{1}{4}$$

Step 4 : If there are initial conditions, impose them to find c_1 & c_2.

Example 2: $y'' + 6y' + 25y = \cos(4x)$

Step 1 : Find general solution to $y'' + 6y' + 25y = 0$.

$y_1(x) = e^{-3x}\cos(4x)$ $\quad y_2(x) = e^{-3x}\sin(4x)$

(see your notes from before).

$$y_h(x) = c_1 e^{-3x}\cos(4x) + c_2 e^{-3x}\sin(4x)$$

Step 2 : Look for $y_p(x) = u_1(x)\, e^{-3x}\cos(4x) + u_2(x)\, e^{-3x}\sin(4x)$

with $u_1'(x)\, e^{-3x}\cos(4x) + u_2'(x)\, e^{-3x}\sin(4x) = 0$.

Skip steps & get to

$$u_1'(x) = -\frac{y_2(x)\, f(x)}{a\,\delta(x)} \qquad u_2'(x) = \frac{y_1(x)\, f(x)}{a\,\delta(x)}$$