Linear differential equations with constant coefficients (continued)

2.1. Variation of parameters (continued)

Example 2: $y'' + 6y' + 25y = \cos(4x)$

$$y_1(x) = e^{-3x}\cos(4x) \quad ; \quad y_2(x) = e^{-3x}\sin(4x)$$

$$y_p(x) = u_1(x)\, e^{-3x}\cos(4x) + u_2(x)\, e^{-3x}\sin(4x)$$

$$u_1' = -\frac{y_2(x)\, f(x)}{a\,\delta(x)} \qquad u_2'(x) = \frac{y_1(x)\, f(x)}{a\,\delta(x)}$$

$$\delta(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = W(y_1, y_2)$$

$$= \begin{vmatrix} e^{-3x}\cos(4x) & e^{-3x}\sin(4x) \\ -3e^{-3x}\cos(4x) - 4e^{-3x}\sin(4x) & -3e^{-3x}\sin(4x) + 4e^{-3x}\cos(4x) \end{vmatrix}$$

$$= e^{-6x}\Big(-3\cos(4x)\sin(4x) + 4\cos^2(4x) - \big(-3\cos(4x)\sin(4x) - 4\sin^2(4x)\big)\Big)$$

$$= e^{-6x}\Big(4\big(\cos^2(4x) + \sin^2(4x)\big)\Big) = 4e^{-6x}$$

$a = 1 \qquad f(x) = \cos(4x)$

$$u_1' = -\frac{y_2(x)\,f(x)}{a\,\delta(x)} = -\frac{e^{-3x}\sin(4x)\cos(4x)}{4\,e^{-6x}}$$

$$= -\frac{1}{4}e^{3x}\sin(4x)\cos(4x)$$

$$= -\frac{1}{8}e^{3x}\sin(8x)$$

$$u_2' = \frac{y_1(x)\,f(x)}{a\,\delta(x)} = \frac{e^{-3x}\cos(4x)\cos(4x)}{4\,e^{-6x}}$$

$$= \frac{1}{4}e^{3x}\cos^2(4x)$$

Integrate :

$$u_1 = \int \frac{-1}{8}e^{3x}\sin(8x)\,dx$$

$$= \int \frac{-1}{8}e^{3x}\,\frac{e^{8ix}-e^{-8ix}}{2i}\,dx = \dots$$

$$= \frac{e^{3x}}{73}\left(\cos(8x) - \frac{3}{8}\sin(8x)\right)$$

$$u_2 = \int \frac{1}{4}e^{3x}\cos^2(4x)\,dx$$

$$= \dots = \frac{e^{3x}}{73}\left(\frac{3}{8}\cos(8x) + \sin(8x)\right) + \frac{e^{3x}}{24}$$

Plug back into expression for y_p:

$$y_p(x) = u_1 e^{-3x}\cos(4x) + u_2 e^{-3x}\sin(4x)$$

$$= \frac{e^{3x}}{73}\left(\cos(8x) - \frac{3}{8}\sin(8x)\right)e^{-3x}\cos(4x)$$

$$+\left(\frac{e^{3x}}{73}\left(\frac{3}{8}\cos(8x) + \sin(8x)\right) + \frac{e^{3x}}{24}\right)e^{-3x}\sin(4x)$$

$$= \left(\cos(8x) - \frac{3}{8}\sin(8x)\right)\cos(4x)\,\frac{1}{73}$$

$$+ \frac{1}{73}\left(\frac{3}{8}\cos(8x) + \sin(8x)\right)\sin(4x)$$

$$+ \frac{1}{24}\sin(4x)$$

$$= \dots = \frac{1}{73}\left(\cos(4x) + \frac{8}{3}\sin(4x)\right)$$

Write general solution

$$y(x) = C_1 e^{-3x}\cos(4x) + C_2 e^{-3x}\sin(4x)$$

$$+ \frac{1}{73}\left(\cos(4x) + \frac{8}{3}\sin(4x)\right)$$