Other types of linear equations

I. Cauchy-Euler equations

This is an equation of the form

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = f(x)$$

Linear, non-homogeneous (if $f \neq 0$), but with non-constant coefficients.

Make the change of variable

$x = e^t$ if $x > 0$	$x = -e^t$ if $x < 0$
$\dfrac{dy}{dx} = \dfrac{dt}{dx}\dfrac{dy}{dt}$	$\dfrac{dy}{dx} = \dfrac{dt}{dx}\dfrac{dy}{dt}$
$dx = e^t dt = x dt$	$dx = \underbrace{-e^t}_{x} dt = x dt$

$$\frac{dt}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt} \iff \text{"}\frac{d}{dx} = \frac{1}{x}\frac{d}{dt}\text{"}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \underbrace{\frac{1}{x}\left(\frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\right)}_{\frac{1}{x^2}\frac{d^2y}{dt^2}}$$