Linear systems (continued)

2. General theory with matrix notation

Consider the 2x2 homogeneous system

$$Y' = M\,Y$$

where M is a matrix with <u>constant real coefficients</u>.

We try a solution in the form

$$Y = \xi e^{\lambda t} \qquad \lambda \in \mathbb{C}$$

ξ = a constant vector $\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \neq 0$

$$Y' = \lambda \xi e^{\lambda t}$$

Substitute into system: $\lambda \xi e^{\lambda t} = Y' = MY = M\xi e^{\lambda t}$

i.e. $\underset{2\times 2}{M}\ \underset{2\times 1}{\xi} = \underbrace{\lambda\,\xi}_{2\times 1}$.

ξ = <u>eigenvector</u> of M
λ = <u>eigenvalue</u> of M

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

$$M\zeta = \lambda\zeta \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \lambda\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$

$$\iff \begin{cases} a\zeta_1 + b\zeta_2 = \lambda\zeta_1 \\ c\zeta_1 + d\zeta_2 = \lambda\zeta_2 \end{cases}$$

2 equations & 3 unknowns : ζ_1, ζ_2 and λ,

$$M\zeta = \lambda\zeta \iff \begin{cases} (a-\lambda)\zeta_1 + b\zeta_2 = 0 \\ c\,\zeta_1 + (d-\lambda)\zeta_2 = 0 \end{cases}$$

$$\iff \underbrace{\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}}_{M-\lambda\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I}}\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= M - \lambda I$$

$$\iff \underbrace{(M-\lambda I)}_{\substack{2\times 2 \\ \text{matrix}}} \underset{\substack{\uparrow \\ 2\times 1 \\ \text{vector}}}{\zeta} = \underset{\substack{\uparrow \\ 2\times 1 \text{ vector}}}{0}$$

For ζ to be non-zero, we need to impose $\det(M-\lambda I) = 0$ i.e. $\lambda^2 - \lambda\, Tr(M) + \det(M) = 0$,