Calculus and Differential Equations II

MATH 250 B

Other types of linear equations

Cauchy-Euler equations

- Cauchy-Euler equations are of the form

$$a_n t^n y^{(n)} + a_{n-1} t^{n-1} y^{(n-1)} + \cdots + a_1 t\, y' + a_0 y = f(t),$$

  where the coefficients a_i are given constants.

- The change of variable $t = e^x$ if $t > 0$ and $t = -e^x$ if $t < 0$ transforms a Cauchy-Euler equation into an equation with constant coefficients for y as a function of x. The resulting equation may then be solved as usual.

- Of course, the general solution $y(x)$ must be re-written in terms of t at the end of the calculation.

- Example: Solve $t^2 \dfrac{d^2y}{dt^2} + 7t\dfrac{dy}{dt} + 25\, y = \cos(4\ln(t)) + \ln(t)$ for $t > 0$.

Higher order equations with constant coefficients

- The methods discussed previously may easily be generalized.
- The characteristic equation associated with

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x)$$

  is a polynomial of order n with real coefficients.
- Because the coefficients are real, the roots are either real or complex conjugate pairs.
  1. A real root λ of multiplicity 1 gives a solution $e^{\lambda x}$.
  2. A real root λ of multiplicity $p > 1$ gives p solutions, $e^{\lambda x}$, $xe^{\lambda x}$, $x^2 e^{\lambda x}, \cdots, x^{p-1} e^{\lambda x}$.
  3. Complex conjugate roots $\lambda = \alpha \pm i\beta$ give solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.
  4. Complex conjugate roots $\lambda = \alpha \pm i\beta$ of multiplicity $p > 1$ give $2p$ solutions, $e^{\alpha x}\cos(\beta x)$, $e^{\alpha x}\sin(\beta x)$, $x\, e^{\alpha x}\cos(\beta x)$, $x\, e^{\alpha x}\sin(\beta x)$, etc.

Higher order equations (continued)

- The method of variation of parameters and the method of undetermined coefficients can also be extended.
- Example: Solve $y''' - 5y'' + 12y' - 8y = e^{3t}\cos(2t) + 7te^t$.
- Variation of parameters: for instance, for a third-order linear equation, if y_1, y_2, and y_3 are three linearly independent solutions to the homogeneous equation, we set $y_p = u_1\, y_1 + u_2\, y_2 + u_3\, y_3$ with

$$\begin{aligned} 0 &= u_1'\, y_1 + u_2'\, y_2 + u_3'\, y_3, \\ 0 &= u_1'\, y_1' + u_2'\, y_2' + u_3'\, y_3', \\ \frac{f(x)}{a(x)} &= u_1'\, y_1'' + u_2'\, y_2'' + u_3'\, y_3''. \end{aligned}$$

  The determinant of this system, which is the Wronskian of y_1, y_2, and y_3, is non-zero since the three solutions are linearly independent.

Other linear equations

For linear equations of order n which do not have constant coefficients and are not Cauchy-Euler equations, try the following.

1. Find (by inspection) one solution to the homogeneous equation.
2. Use the method of reduction of order repeatedly to find n linearly independent solutions to the homogeneous equation.
3. Write the general solution to the homogeneous equation, y_h, as a linear combination of the above n linearly independent solutions.
4. Find a particular solution y_p (by inspection, by the method of undetermined coefficients, or by variation of parameters).
5. Write the general solution, $y = y_h + y_p$.
6. Apply initial or boundary conditions, if any.