Math 322 **Final Exam**

Name:

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☐ Check box if you are part of a study group

Notes:

1. Show all of your work, present it neatly, and explain what you are doing. In particular, write complete sentences, either using words or mathematical symbols.
2. **You will only receive credit for the work that is shown.**
3. It is **more important** to leave out one or two sub-questions and do all of the others in depth, than to do a little bit of each and finish none.
4. **Two pages of formulas** for Fourier series and transforms and for Laplace transforms are attached to this exam.
5. **Question 9 is a bonus question and is worth 10 points.**

Question	1	2	3	4	5	6	7	8	**Total**
Score	/12	/15	/20	/20	/10	/35	/23	/25	**/160**

1) [12 points] Find all of the values of z for which $z^6 = -64$. Show all your work. Give an exact, but simplified, answer for each root.

2) Consider the function $u(x, y) = \sin(x)\cosh(y)$, where x and y are real variables.

   a) [4 points] Show that the function $u(x,y)$ is harmonic. Show all your work.

   b) [8 points] Find a function $v(x,y)$ such that u and v are harmonic conjugates. Explain what you are doing.

c) [3 points] Write $u(x,y) + i\, v(x,y)$ as a function of $z = x + i\, y$. Show all your work.

3) [20 points] Solve the following system of differential equations

$$\begin{cases} \dfrac{dx_1}{dt} = 3x_1 - x_2 \\ \dfrac{dx_2}{dt} = 8x_1 - 3x_2 \end{cases},$$

with initial conditions $x_1(0) = 2$, $x_2(0) = 3$.

You may use the method of your choice, but you have to show all of your work. If you use Laplace transforms, remember that $\mathcal{L}\left(e^{at}\right)(s) = \dfrac{1}{s-a}$.

4) [20 points] Find the Fourier transform of $f(x) = x\, e^{(-a|x|)}$, where $a > 0$. Show all your work.

5) [10 points] Consider the following polynomials (they are two Legendre polynomials):

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \qquad P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right).$$

Calculate the dot product of P_2 and P_3, and show that these two polynomials are orthogonal. Here, the dot product of two functions u and v is defined as

$$\langle u, v \rangle = \int_{-1}^{1} u(x)\, v(x)\, dx.$$

6) [35 points] Solve the heat equation,

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0, \qquad 0\le x\le 2, \qquad t\ge 0,$$

with boundary conditions $u(0,t) = 0$ and $u(2,t) = 0$, and initial condition $u(x,0)=\sin(\pi x)+7\sin(2\pi x)-8\sin(4\pi x)$. Clearly identify all of the steps in solving this equation. Show all your work.

7) Consider the following vectors in $\mathbb{R}^3$.

$$U_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \; U_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \; U_3 = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}.$$

a) [3 points] Show that U_1, U_2 and U_3 are orthogonal to one another.

b) [10 points] Show that U_1, U_2 and U_3 form a basis of $\mathbb{R}^3$. Explain your reasoning.

c) [10 points] Expand the vector $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ on the orthogonal basis $\{U_1, U_2, U_3\}$. In other words, write X as a linear combination of U_1, U_2 and U_3. Show all your work.

8) Consider the function $f(x) = e^x$, defined on the interval $-1 \le x \le 1$.

a) [17 points] Find the **complex** Fourier series of the function *f*. Show all your work.

b) [8 points] What is the Fourier series of *f* equal to at $x = 1$? Explain.

9) **Bonus questions (10 points):** Calculate the Laplace transform of $f(t) = t$ and use this information to find the inverse Laplace transform of $\frac{1}{s^4}$.

Generalize your result and find the inverse Laplace transform of $\frac{1}{s^n}$, for $n = 2, 3, 4, \ldots$.