Chapter 6: Laplace Transforms

1. Definitions

- The Laplace transform, \( \mathcal{L}(f) \), of a piecewise continuous function \( f \) (defined on \([0, \infty)\)) is given by

\[
\mathcal{L}(f)(s) = F(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt.
\]

- Clearly, the above integral only converges if \( f \) does not grow too fast at infinity. More precisely, if there exist constants \( M > 0 \) and \( k \in \mathbb{R} \) such that

\[ |f(t)| \leq M \exp(kt) \]

for \( t \) large enough, then the Laplace transform of \( f \) exists for all \( s > k \).

- If \( f \) has a Laplace transform \( F \), we also say that \( f \) is the inverse Laplace transform of \( F \), and write \( f = \mathcal{L}^{-1}(F) \).

2. Properties of the Laplace transform

- The Laplace transform is a linear transformation, i.e. if \( f_1 \) and \( f_2 \) have Laplace transforms, and if \( \alpha_1 \) and \( \alpha_2 \) are constants, then

\[
\mathcal{L}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{L}(f_1) + \alpha_2 \mathcal{L}(f_2).
\]

- As for Fourier transforms, the statement

\[
f = \mathcal{L}^{-1}\left(\mathcal{L}(f)\right)
\]

should be understood in a point-wise fashion only at points where \( f \) is continuous.

- Since there is no explicit formula for the inverse Laplace transform, formal inversion is accomplished by using tables, shifting \( t \) and \( s \), taking derivatives of known Laplace transforms, or integrating them.

- Note: All of the formulas written in what follows implicitly assume that the various functions used have well-defined Laplace transforms. One should therefore check that the corresponding Laplace transforms exist before using these formulas.

- \( s \)-shifting formulas

\[
\mathcal{L}\left(e^{at}f(t)\right)(s) = F(s-a), \quad e^{at}f(t) = \mathcal{L}^{-1}(F(s-a))(t).
\]

- Laplace transform of derivatives

\[
\mathcal{L}\left(f'\right)(s) = s \mathcal{L}(f)(s) - f(0),
\]

\[
\mathcal{L}\left(f''\right)(s) = s^2 \mathcal{L}(f)(s) - sf(0) - f'(0).
\]
Laplace transform of derivatives and antiderivatives

- More generally,
  \[ \mathcal{L} \left( f^{(n)} \right)(s) = s^n \mathcal{L}(f)(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0). \]

- Laplace transform of antiderivatives
  \[ \mathcal{L} \left( \int_0^t f(\tau) \, d\tau \right)(s) = \frac{1}{s} \mathcal{L}(f)(s), \]
  \[ \int_0^t f(\tau) \, d\tau = \mathcal{L}^{-1} \left( \frac{1}{s} \mathcal{L}(f)(s) \right)(t). \]

Examples:
- Find the Laplace transforms of \( \sin(\omega t) \) and \( \cos(\omega t) \).
- Find the inverse Laplace transforms of \( 1/(s(s^2 + 1)) \) and \( 1/(s^2(s^2 + 1)) \).

Heaviside and delta functions; \( t \)-shifting

- The Heaviside function (or step function) \( H(t) \) is defined as
  \[ H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}. \]

- We can calculate that, for \( a > 0 \),
  \[ \mathcal{L}(H(t - a))(s) = \frac{e^{-as}}{s}. \]

- More generally, for \( a > 0 \), we have the following time-shifting formulas
  \[ \mathcal{L}(f(t - a)H(t - a))(s) = e^{-as} \mathcal{L}(f)(s) \]
  \[ f(t - a)H(t - a) = \mathcal{L}^{-1}(e^{-as} \mathcal{L}(f)(s))(t). \]

- The above formulas are useful to calculate the Laplace transforms of signals that are defined in a piecewise fashion.

Delta functions

- The Dirac delta function (or distribution) is defined as the limit of the following sequence of narrow top-hat functions,
  \[ \delta(t) = \lim_{\epsilon \to 0} f_\epsilon(t), \quad f_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon} & \text{if } |t| \leq \epsilon \\ 0 & \text{otherwise} \end{cases}. \]
  Since \( \int_{-\infty}^{\infty} f_\epsilon(t) \, dt = 1 \), we also write that \( \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \).

- More generally, for a “well-behaved” function \( g \), we have
  \[ \int_{-\infty}^{\infty} g(t) \delta(t - a) \, dt = g(a). \]

- For \( a > 0 \), this allows us to define the Laplace transform of \( \delta(t - a) \) as
  \[ \mathcal{L}(\delta(t - a))(s) = e^{-as}. \]

Differentiation and integration of Laplace transforms

In what follows, we write \( \mathcal{L}(f)(s) \) as \( F(s) \).

- Differentiation of Laplace transforms
  \[ \mathcal{L}(t f(t))(s) = -F'(s), \quad \mathcal{L}^{-1}(F'(s))(t) = -t f(t). \]

- Integration of Laplace transforms
  \[ \mathcal{L} \left( \int_s^\infty F(\nu) \, d\nu \right)(s) = \frac{f(t)}{t}, \quad \mathcal{L}^{-1} \left( \int_s^\infty F(\nu) \, d\nu \right)(t) = \frac{f(t)}{t}. \]

- Example: Find the inverse Laplace transform of \( s/(s^2 + 1)^2 \).
Applications to ODEs and systems of ODEs

- Solve $y'' + y = t/\pi$, with $y(\pi) = 0$ and $y'(\pi) = 1 + 1/\pi$.

- Let $f(t) = \begin{cases} \frac{1}{2\pi} & \text{if } 1 - \epsilon \leq t \leq 1 + \epsilon \\ 0 & \text{otherwise} \end{cases}$, where $\epsilon < 1$. Solve $y'' + 4y' - 5y = f(t)$ with initial conditions $y(0) = 0$ and $y'(0) = 0$.

- Solve $y'' + 4y' - 5y = \delta(t - 1)$, with initial conditions $y(0) = 0$, $y'(0) = 0$.

- Solve the initial value problem $\frac{dX}{dt} = AX$, 

  \[
  A = \begin{bmatrix} -13 & -36 \\ 6 & 17 \end{bmatrix}, \quad X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
  \]