Math 322 **Midterm I**

Name:
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Notes:

1. Show all of your work, present it neatly, and explain what you are doing. In particular, write complete sentences, either using words or mathematical symbols.
2. **You will only receive credit for the work that is shown.**
3. You can score a maximum of 130 points on this test. So **there are 10 "bonus" points**.
4. It is **more important** to leave out a problem and do all of the others in depth, than to do a little bit of each problem and finish none.

Question	1	2	3	4	5	6	7	8	9	**Total**
Score	/20	/20	/20	/25	/5	/10	/10	/10	/10	**/120**

1. Consider the function $f(z) = \dfrac{i}{z-1}$.

a) [5 points] Write down the Cauchy-Riemann equations for an analytic function $u(x,y) + i\, v(x,y)$.

b) [12 points] Use the Cauchy-Riemann equations to show that the function f above is analytic everywhere, except at $z = 1$.

c) [3 points] Is the function f entire? Why or why not?

2. Consider the function $u(x,y) = x\,(x^2 - 3\,y^2)$.

a) [5 points] Show that u is harmonic.

b) [12 points] Find the harmonic conjugate of u, and call this function $v(x,y)$. Explain what you are doing.

c) [3 points] Express $u(x,y) + i\,v(x,y)$ as a function of the variable $z = x + i\,y$. Show all of your work.

3. Evaluate the following expressions and write the result in the form $a + i\,b$, where a and b are real. Give **exact answers** (do not approximate them numerically). If a function is multi-valued, give **all of the possible values** of that function.

a) [3 points] $\mathrm{Ln}\left(\sqrt{3}+i\right)$

b) [3 points] $\ln\left(\sqrt{3}+i\right)$

c) [3 points] $\cosh(2+i\pi)$

d) [3 points] $\exp(7i\pi+1)$

e) [8 points] $\sqrt[3]{-2+2i}$

4. Consider the function $g(z) = \frac{1}{\overline{z}}$.

a) [4 points] Given that it involves a term in $\overline{z}$, do you expect $g(z)$ to be differentiable at $z = 1$? Why or why not?

b) [4 points] Express the following difference quotient in terms of Δx, and Δy: $Q(\Delta x, \Delta y) = \frac{g(1+\Delta z) - g(1)}{\Delta z}$, where $\Delta z = \Delta x + i\,\Delta y$. **Do not try to separate the real part of $\boldsymbol{Q}$ from its imaginary part.**

c) [10 points] Calculate the limit of $Q(\Delta x, \Delta y)$ as $\Delta z \to 0$, along the line of equation $\Delta y = t\,\Delta x$. Show all your work.

d) [4 points] Using the result of part c), show that the function $g(z)$ is not differentiable at $z = 1$. Justify your answer.

e) [3 points] Is the function g analytic at z = 1? Why or why not?

5. [5 points] Give an example of a multi-valued function of a complex variable. Explain why it is multi-valued.

6. [10 points] Use De Moivre's formula to express $\cos(4\,x)$ in terms of $\cos(x)$ and $\sin(x)$.

7. [10 points] Is the following set of vectors linearly independent? Why or why not? $\left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

8. [10 points] Find a basis of the column space of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 2 & 1 \\ 7 & 5 & 1 \end{bmatrix}$.

Note that the columns of A are the same vectors as in Problem 7 above. Explain what you are doing.

9. Consider the vector $U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

a) [5 points] Find two vectors, U_1 and U_2, whose entries are **all non-zero**, and which, together with the vector U_3 above, form a basis for $\mathbb{R}^3$.

b) [5 points] Explain why the set $\{U_1, U_2, U_3\}$ that you have found is a basis of $\mathbb{R}^3$.